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Michaellike Spaces: Semi-Topological Groups, Cancellation and Quotients

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Abstract

The Michael line is a particular case of a topological space obtained from a pair (X, Y), where X is a topological space, and Y a subset. The space obtained with full generality is what we call Michaellike type space below.

We collect here results on these spaces having to do with compactness, connectedeness, group structures, the cancellation problem and quotients.

The section on quotients is motivated by results in [1], where we showed that some minimal universal spaces can be obtained as quotients of Michaellike spaces starting with the real line.

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1. Compactness

Let X be a topological space and Y be a proper subset of X.

Definition 1.1. The *Michaellike space* X_Y is the topological space which has X as underlying set and whose open sets are the unions of the open sets of X with subsets of Y.

For a Michaellike space X_Y, we will denote by p_Y the projection onto a quotient X_Y/G, where G is any homeomorphism group.

Definition 1.1 is, naturally, motivated by [4] and such spaces are mentioned, for instance, in [3].

Let us take a Michaellike space X_Y and let us see how it is related to the spaces Y and $X \setminus Y$ in terms of compactness.

Observe first that if X_Y is compact then the same happens with $X \setminus Y$ for this space is closed in the former.

If X_Y and $X \setminus Y$ are both compact it does not follow that Y is compact.

Let X be a non-finite set and Y a proper non-finite subset. Fix $p \in Y$ and take for topology in X the collection formed by the empty set, X and the subsets of Y that contain p. It follows that X_Y and $X \setminus Y$ are both compact and yet Y is not compact.

Finally it is also the case that if $X \setminus Y$ and Y are compact then X_Y is not necessarily compact.

Let X be a non-finite set and $Y = X \setminus \{p_1, \dots, p_k\}$. Take Y and the singletons $\{p_i\}, i = 1, \dots, k$, for basic open sets in X. Then $X \setminus Y$ and Y are compact but X_Y is not since it is a discrete infinite space.

But we have

Proposition 1.2. Let X be a topological space and Y a proper subspace. If $X \setminus Y$ is compact and Y is finite then X_Y is compact.

Proof. Let $(U_i), i \in I$, be an open covering for X_Y. Each U_i is a union $X_i \cup Y_i$, where X_i is open in X and Y_i is a subset of Y. From the X_j's we get a finite covering of $X \setminus Y$. If we add to the corresponding U_j's a finite number of U_k's whose union contains Y we obtain a finite subcovering of the initial covering. ■

2. Connectedness

When dealing with Michaellike spaces, connectedness questions only arise if the initial space is not T_1. If X is T_1 then X_Y has a singleton at least that is both open and closed.

If we take X_Y, Y and $X \setminus Y$ the connectedness of any two of them does not imply the third is connected. Let $X = \{a, b, c, d\}$.

If we take $\{a\}, \{a, b\}, \{a, c\}, \{a, d\}$ for basic open sets of X and let $Y = \{b\}$ then we have a topological space which is not T_1 such that Y, $X \setminus Y$ are connected and X_Y is not connected.

If now we take $\{a, b\}, \{a, b, c\}, \{a, b, d\}$ for basic open sets of X and let $Y = \{a, b\}$ then we have a topological space which is not T_1 such that X_Y, Y are connected and $X \setminus Y$ is not connected.

Finally take as basic open sets X, $\{a, b\}, \{c\}$ and let $Y = \{a, b, c\}$. Then X_Y, $X \setminus Y$ are connected but Y is not.

Proposition 2.1. Let X be a connected topological space which is not T_1. Then there is $p \in X$ such that $X_{\{p\}}$ is connected.

Proof. If X is not T_1 there is $p \in X$ such that $\{p\}$ is not closed. Hence there is $q \in X$ such that any open set U_q, containing q, contains p.

Assume that $X_{\{p\}}$ is not connected and that we have $X_{\{p\}} = A \cup B$, where A and B are open, disjoint sets. If $q \in A$ then A is an open set in X containing p. Consequently, since B is open in $X_{\{p\}}$ and is contained in $X \setminus \{p\}$, B is open in X and we have a disconnection for this space. ■

The point mentioned in this proposition may or may not form an open singleton in the original space.

It may also happen that there is no $Y \neq \{p\}$ such that X_Y is connected. An example is provided by $X = \{a, b, c, d\}$, with $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$ as basic open sets.

Moreover, as can be seen from the proof, $X_{\{p\}}$ is connected for any singleton $\{p\}$ which is is not closed.

3. Semi-topological groups

A *semi-topological group* is a group and a topological space G such that multiplication on the left L_a and multiplication on the right R_a are continuous maps, for every $a \in G$.

Below one assumes the same group operation in G and G_Y.

Proposition 3.1. Let G be a topological space which is also a group. Then there is no proper subset Y such that G_Y is a non-discrete semi-topological group.

Proof. Let Y be such that G_Y is a non-discrete semi-topological group. Then there is $x \in G \setminus Y$ such that $\{x\}$ is not open. Choose $y \in Y$ and let z be such that $xz = y$. Any open set containing x contains, at least, another point x'. Since $\{y\}$ is open, $R_z : G_Y \to G_Y$ is not continuous. ■

Proposition 3.2. Let G be a semi-topological group and Y be a proper subset. If G_Y is a semi-topological group then G is discrete.

Proof. If G_Y is a semi-topological group then it is discrete. All the points in $G \setminus Y$ give rise to open singletons in G. If G is not discrete then there is $x \in Y$ such that $\{x\}$ is not open. Choose $y \in G \setminus Y$ and let z be such that $xz = y$. As above, it follows that $R_z : G \to G$ is not continuous. ■

Proposition 3.3. Let G be a connected, T_1 topological space which is an algebraic group. Then there is no proper subset Y such that G_Y is a semi-topological group.

Proof. Assume that Y is such that G_Y is a semi-topological group. Since L_a is a homeomorphism for every $a \in G$, we must have $L_a(Y) \subset Y$, from where it follows that $Y \subset L_{a^{-1}}(Y)$. Consequently, we also have $Y \subset L_a(Y)$, for every $a \in G$. Hence $Y = L_a(Y)$.

If one takes an element $b \in Y$ and one considers $L_{b^{-1}}$ we see that the identity element is in Y. This implies that $a \in Y$, for every $a \in G$, and $Y = G$. ■

4. Cancellation

The cancellation problem, which goes back to S. Ulam [2], is formulated as follows:

Given topological spaces X, Y, Z, under what circumstances does $X \times Z \approx Y \times Z$ ($\approx$ meaning homeomorphic to) imply $X \approx Y$?

Proposition 4.1. Let Y be a subset, with more than one point, of the T_1 topological space X such that $X \setminus Y$ is connected. If Z_1, Z_2 are connected and $X_Y \times Z_1 \approx X_Y \times Z_2$ then $Z_1 \approx Z_2$.

Proof. Let $p \in Y$. Then the singleton $\{p\}$ is open and closed in X_Y and the same happens with $\{p\} \times Z_1$. Hence this latter subset is a component of $X_Y \times Z_1$.

Take now $(X \setminus Y) \times Z_2$ and observe that its subset topology is the same whether we consider it a subspace of $X \times Z_2$ or of $X_Y \times Z_2$. Hence it is connected. Since it is maximal, it is a component of $X_Y \times Z_2$.

If ϕ denotes a homeomomorphism between $X_Y \times Z_1$ and $X_Y \times Z_2$, we have $\phi(\{p\} \times Z_1) = \{q\} \times Z_2$, in which case $Z_1 \approx Z_2$, or $\phi(\{p\} \times Z_1) = (X \setminus Y) \times Z_2$. If the latter were the case, choose $r \in Y \setminus \{p\}$. We would then have $\phi(\{r\} \times Z_1) = \{s\} \times Z_2$, for some $s \in Y$ which, again, would imply $Z_1 \approx Z_2$. ■

5. Quotients

We get here results concerning three of the four 2-point spaces involving qotients of Michaellike spaces.

a) Starting with T_1 spaces

We assume that X will denote a connected, T_1, topological space unless we state otherwise.

We start with the following observation. If G is the homeomorphism group of the Michaellike space X_Y, then Y is an equivalence class for the equivalence relation induced by the action of G.

In fact, if $y_1, y_2 \in Y$ we may take the homeomorphism which interchanges those two points and that fixes any point outside $\{y_1, y_2\}$. On the other hand no point $x \notin Y$ is related to a point $y \in Y$. If that happened the singleton $\{x\}$ would be simultaneously open and closed in X.

Proposition 5.1. Let G be the homeomorphism group of X_Y. If X is connected and T_1 and X_Y/G is discrete then Y is closed in X.

Proof. Above we saw that $Y = [y]$, for any $y \in Y$. If X_Y/G is discrete then $p_Y^{-1}(X_Y/G \setminus \{[y]\})$ is open in X_Y and also in X. Hence Y is closed. ■

The converse of Proposition 5.1 is false. Let X be the interval $[0, 1)$ with the standard topology. Let $Y = \{p\}$, where p is non-zero. Then X_Y/G is not discrete. In fact, it happens that $[0] = \{0\}$.

However we have

Proposition 5.2. Let G be the homeomorphism group of X_Y. If X is locally connected, connected and T_1, Y is closed and $X \setminus Y$ is homogeneous then X_Y/G is a 2-point discrete space.

Proof. Let $y_1, y_2 \in X \setminus Y$ and assume that they belong to components C_1 and C_2, respectively, of $X \setminus Y$. Since $X \setminus Y$ is homogeneous there is $h : X \setminus Y \to X \setminus Y$ such that $h(y_1) = y_2$ and hence $h(C_1) = C_2$. Now using h, h^{-1}, if $C_1 \neq C_2$, and the identity map outside $C_1 \cup C_2$, we obtain an element of G that maps y_1 to y_2.

Then X_Y/G is formed by two elements, whose inverse images by p_Y are Y and $X \setminus Y$ and, consequently, it is a discrete space. ■

If we drop the connected assumption for X then we cannot rule out the possibility of having a point in Y related to a point in $X \setminus Y$. If that happened then X_Y itself would be discrete and the quotient X_Y/G would have just one point.

We could replace in the statement of Proposition 5.2 the condition $X \setminus Y$ *is homogeneous* by the weaker one *each component of* $X \setminus Y$ *is homogeneous*. Then we would also obtain a discrete space and the inverse images by p_Y would be Y and the unions of the homeomorphic components of $X \setminus Y$.

Proposition 5.3. Let G be the homeomorphism group of X_Y. Assume that X is connected and T_1. If X_Y/G is the Sierpinski space then $X \setminus Y$ is not open in X.

Proof. If X_Y/G is homeomorphic to the Sierpinski space then there are just two equivalence classes, of which one is Y and the other one its complement. Since Y is open in X_Y it follows that $X \setminus Y$ is not and, consequently, $X \setminus Y$ is not open in X either. ■

Again the condition is necessary but not sufficient. If $X = [0, 1)$ and $Y = (p, 1)$, $p \neq 0$, then X_Y/G is a 4-point space.

Proposition 5.4. Let G_1 be a group of homeomorphisms of X and Z be a G_1-orbit which is not open in X. Take $Y = X \setminus Z$ and denote by G the homeomorphism group of X_Y. If X is connected and T_1 then X_Y/G is the Sierpinski space.

Proof. If $g \in G_1$ then $g \in G$ since $g(Z) = Z$. By the observation made above, at the beginning of this section, X_Y/G then consists of just two equivalence classes, Y and $Z = X \setminus Y$. Since Y is open in X_Y and Z is not, it follows that X_Y/G is homeomorphic to the Sierpinski space. ■

It remains to observe that no 2-point trivial space can be obtained as the quotient of a Michaellike space. That follows directly from the fact that such a quotient has an open singleton, the image $p_Y(y)$, for any $y \in Y$.

b) Starting with spaces which are not T_0

It is possible to obtain the Sierpinski space and the discrete 2-point space starting with topological spaces which are not even T_0.

Assume that X is a trivial topological space and that Y is a proper subset. Then the open sets in X_Y are X and the subsets of Y. If G is the homeomorphism group of X_Y then X_Y/G is the Sierpinski space.

Let now X be a topological space with only one proper open set Y_1. Denote by Y the complement of Y_1 and let G be the homeomorphism group of X_Y. Then X_Y/G is the discrete space.

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