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On Zero-Divisor Graphs of Modules

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Abstract

Here we investigate some graph theoretic aspects of modules with the help of the attached ring through various types of maps, in particular module homomorphisms from the module to it. Here we especially dealt with the cases of finite dimensional modules with ascending chain conditions with its annihilators.

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1. Introduction

Here we investigate some graph theoretic aspects related to coloring of elements of modules over a ring introducing the notion of adjacency in between its elements.

In an additive algebraic structure, a product operation is may arise with the help of another structure equipped with two operations, one of them bearing the nature of so-called product. And the same may be obtained when we be able to give an external composition to the preceding algebraic structure with the help of the later one. If M is with such an additive structure and A is the other one with operation '•', at least bearing the semi-group character such that we have the map: $M \times A \to M$ with $(m,a) \to m \cdot a$ (we will write it as just ma), then for a map $f : M \to A$, it is possible to define a product '$\bullet_f$' such that for $a,b \in A, a \bullet_f b = a.f(b)$ [or just $af(b)$].The system $(M,+,\bullet_f)$ is now our sought structure for having a system with the chosen M with a pseudo product $\bullet_f$. For such an algebraic structure our expected graph is the triplet (V,E,f) with V as the set of vertices-the elements of M, E-the set of edges where for $a, b(\in M)$, 'a is adjacent to b' if $a \bullet_f b = 0$ (or '*a and b are f- adjacent').* In other word this associated graph to the module M over R is the directed zero divisor graph of the structure $(M, +. \bullet_f)$ if we consider two non zero element a and b are adjacent with the above definitions.

The pseudo product described here may coincide with the product available in R when M coincides with the ring R with respect to some definite map f (in particular the identity map). It is observed here some interesting and elegant connections between finiteness of the chromatic number of such a graph and the notion of a so-called Goldie module [4] which reads as a module M over a ring R where,

1. M is finite dimensional
2. R satisfies the ascending chain condition (a.c.c.) for annihilators of subsets of M in R .i.e. any collection of annihilators of subsets of M in R satisfies the maximal condition.

Here we give some interesting facts revealing the interrelation between such a module and the corresponding f - algebraic triplet, when the latter structure appears as a result of an attachment of respective order preserving map f .In this context we would like to refer Chartr and et al [3] and Pilz [8] for Graph Theoretic and Near-ring theoretic pre-requisites. Moreover we refer the works of Beck [1], Bruijn et-al [2] for coloring of rings.

If M is a right module over R (denoted M_R) and if $f : M \to R$ is an R-homomorphism, then the triple $(M, +, \bullet_f)$ where $m_i \bullet_f m_j = m_i f(m_j)$ is a ring. We call it an *h-ring* denotes M_f .

Note

1. This ring is our sought *h-ring.*
2. Such an *h-ring* is not commutative always. We see that in case of a PID D if $f : D \to D$, a D-homomorphism with (say) $f(x) = mx, (m \in D)$, then D_f is commutative. It happens to be true for any such f .
3. Even if the ring R does not contain unity 1, yet when M is an R-module, M_f may contain its unity. The following explanation justifies what we have meant.

Let $R = \{(2m, 2n)/ m, n \in Z\}$.Then $(R, +, \cdot)$ is a commutative ring and is without unity such that for $(2m, 2n), (2a, 2b) \in R$, $(2m, 2n) + (2a, 2b) = (2(m + a), 2(n + b))$ and $(2m, 2n) \cdot (2a, 2b) = (2ma, 4nb)$

Now $(Z, +)$ is a module over R with the map $Z \times R \to Z$ where $m(2\alpha, 2\beta) = m\alpha \in Z$ It is easy to see that for $m, n \in Z$ and $(2,2n) \in R$, $m(2,2n) = m$ .i.e. R has no unity, but $(2,2n) = e \in R$ is such that $me = m \forall m \in Z$ .Thus e acts as a right unity for the module Z over the ring R .For any $n \in Z$ it is true. Now $f : Z \to R$

with $f(m)=(2m,0)$ is an R-homomorphism. And for $1\in Z$, we have $m\bullet_f 1 = mf(1)=m(2,0)=m\forall m\in Z$ .Thus $1\in Z$ acts as the unity in $(Z)_f$

4. When $f=I$ -the identity map and the attached ring is commutative then the *h-ring* M_f is also so.

Unless otherwise specified, through out this paper M (or M_R) will mean a right module over R and f is an R-homomorphism from M to R. A module M_R is an *f-color- module* if the *f-chromatic number* [§ 3] of the module M [denoted $\chi_f(M)$] is finite. The main results of this paper are dealt here from two main aspects viz. *h-ring* and that of coloring of modules such as:

A sub-module of the module appears as a right ideal of the corresponding *h-ring* structure of the same. The *h-ring* structure of a module M_R gives rise to the *h-ring* structure of the quotient of M modulo an ideal contained in the kernel of the map- f induced by the preceding map f .A module M_R is right Goldie if and only if the corresponding *h- ring* is right Goldie. Prime character of a sub-module modulo the kernel of the homomorphism leads to the prime character of the image sub-module of the ring as a module over itself. Pseudo product defined with the help of just a map f from the module M_R to the ring R gives rise to an abelian near-ring structure. A group structure is sufficient to make it into a right near ring with respect to the *pseudo- product* defined. Occurrence of infinite number of *f-right finite* elements in a module M_R is a sufficient condition for having an infinite *f-clique* in it. An infinite *f-clique* in an *h-ring* helps to get such an induced clique in the quotient modulo the kernel and conversely. The existence of an infinite *f-clique* in a module is a necessary condition for containing a nilpotent element in the *h-ring* M_f .A finite dimensional module turns out to be a right Goldie *h- ring* if it is with only finite *f-cliques*. The prime sub-module [§3] of a Goldie module M_R and one that of the attached ring R annihilates each other if the intersection of the respective annihilators vanishes. An *h-ring* originated from a finite dimensional *f-color module* is a right Goldie ring. In case of an *f-color-module* M_R, the *h-ring* M_f modulo right annihilator of any subset of M is a *ψ-color-module* (ψ is induced by f).

2. Examples and Observations

The following examples given below in this paper carries informations about nearing groups, semi Ring groups or such type of algebraic structures viz: Goldie M -group [4] in the context of the results stated.

Example 1: Consider the module $Z[x]$over the ring Z .Define $f: Z[x]\to Z$ by

$f\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i$, $a_i \in Z$ then $(Z[x],+,\bullet_f)$ is a ring. Hence $Z[x]$ is an *h-ring.*

Example 2: $Z(x)$ is a module over Z. Define $f : Z[x] \to Z$ by $f(p(x)) = \deg(p(x)), p(x) \in Z(x)$, then $(Z[x],+,\bullet_f)$ is a right near- ring. Hence $Z[x]$ is an *f- right near-ring.*

Example 3: $Z_{30}[x]$ is an additive abelian group of polynomials over Z_{30}. Then $(Z_{30}[x],+,\circ)$ is a right nearing where'$\circ$ 'is defined as the composition of mappings. Then $I_6[x]$ is an additive subgroup of $Z_{30}[x]$ where $I_6 = \{0,6,12,18,24\}$ .Let N = $(Z_{30}[x],+,\circ)$ and G $=(I_6[x],+)$.Now G is an N -subgroup of N. Define $\phi : G \to N$ by $\phi(g) = 2g$ .Then $(G,+,\bullet_\phi)$ is a right near-ring. Thus G is a ϕ*-right near-ring.*

Example 4: Consider the semi group $(A,\cdot)$ where $A = \left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix}\right\}$.

Let $A_0 = \begin{pmatrix}0&0\\0&0\end{pmatrix}, A_1 = \begin{pmatrix}1&0\\0&1\end{pmatrix}, A_2 = \begin{pmatrix}0&1\\1&0\end{pmatrix}, A_3 = \begin{pmatrix}1&1\\1&1\end{pmatrix}$.Here

$\cdot$	A_0	A_1	A_2	A_3
A_0	A_0	A_0	A_0	A_0
A_1	A_0	A_1	A_2	A_3
A_2	A_0	A_2	A_1	A_3
A_3	A_0	A_3	A_3	A_0

And $(Z_2,+,\cdot)$ is a ring. We define $f : A \to Z_2$ by $f(A) = (a_{11})$ where $A = (a_{ij})$ .Here $\bullet_f$ is right distributive but not left distributive. Hence $(A,\cdot,\bullet_f)$ is a *semi near-ring* (right).Thus A is *f-semi near-ring* (right). If we define $f : A \to Z_2$ by $f(A) = \det(A)$ and $f(A) = traceA$ or $f(A) = \sum a_{ij}$ then $(A,\cdot,\bullet_f)$ is a *semi near-ring* (right) and *semi-ring* respectively.

Example 5: Let R be a ring. Then R is a module over itself. Consider a map $f : R \to R$ defined as $f(x) = 2x \forall x \in R.$ then $\bullet_f$ is both left and right distributive. Thus $(R,+,\bullet_f)$ is a ring. Hence R is an *h-ring*.

Note: In a given ring we can insert a number of rings.

Example 6: We consider $R = z_6 = \{0,1,2,3,4,5\}$ and $f : R \to R$ by $f(x) = 2x$. Then $(R,+,\bullet_f)$ is a ring. We need only three colors to color the ring $(R,+,\bullet_f)$.Thus $\chi_f(R) = 3$.If we define $g : R \to R$ by $g(x) = 3x$ then only four colors are needed to color the ring $(R,+,\bullet_g)$.Thus $\chi_g(R) = 4$.If we consider the identity map $i : R \to R$, we need three colors to color the ring $(R,+,\bullet_i)$.Thus $\chi_i(R) = 3$.This $\chi_i(R) = 3$ is nothing but the $\chi(R)$ in case of Beck[1].

Example 7: Consider the group $G = \{0, a, b\}$ under the addition defined by the following table.

+	0	a	b
0	0	a	b
a	a	b	0
b	b	0	a

And let the set $N = M(G) = \left\{\alpha \mid \alpha : G \xrightarrow{map} G\right\}$ We define $\phi : G \to N$ by

.	0	a	b
$\phi(0) = \alpha_1$	0	0	a
$\phi(a) = \alpha_2$	0	a	0
$\phi(b) = \alpha_3$	a	0	0

Then only two colors are needed to color the vertices of $(G,+,\bullet_\phi)$. Thus $\chi(G,+,\bullet_\phi)$ is 2. If $\phi : G \to N$, defined by

.	0	a	b
$\phi(0) = \alpha_1$	0	0	0
$\phi(a) = \alpha_2$	a	a	a
$\phi(b) = \alpha_3$	b	b	b

Then $(G,+,\bullet_\phi)$ is a left near-ring. Here $\chi(G,+,\bullet_\phi)$ is 1.

3. Preliminaries

Definitions

The graph of the module M is the triplet (V, E, f) where the elements of the module M is the set of vertices V, E is the set of edges and what follows f is an R-

homomorphism from a module M_R to the module R_R. An element $x(\in M)$ is f-*adjacent* to y if $x \bullet_f y = 0$ or we call it "x and y are f-*adjacent*" (clearly such a graph is a directed one).Two elements "x and y are f-*adjacent* to each other" if $x \bullet_f y = 0 = y \bullet_f x$. If x and y are not distinct then (x, x) is an f-*loop*. A subset C (finite /infinite) of M is an f-*clique* if any two elements of C are f-*adjacent* to each other. The minimum numbers of colors which can be assigned to the elements of M so that no two f-*adjacent* elements have same color is the f-*chromatic* (denoted $\chi_f(M)$) number of M. A module M is an f-*color module* if the f-*chromatic* number of the module M is finite. If $\chi_f(M) = n$, whatever be the choice of the R-homomorphism $f(\neq 0)$, then n is the chromatic number of the module M denoted $\chi(M) = n$.

A module M is *color-module* if $\chi(M)$ is finite.

One may interestingly note here that chromatic number of a ring, what Beck has insisted [1] need not be unique. The $\chi(R)$ in the sense what Beck has meant is nothing but our $\chi(R_i)$ where $i : R \to R$ is the identity homomorphism. Example7 states what we have mentioned here.

A module M_R is said to be *prime* if for any sub-module N_R of M, $Ann(M_R) = Ann(N_R)$ .An element x in M is f-*left-finite* (f-*right- finite*) if $M \bullet_f x$ ($x \bullet_f M$) is finite. A ring R is *reduced* if *nil radical* $J = (0)$. It is to be noted that in case of a non-commutative ring R, for every $r \in R, rR = R$, then R is a division ring. On the other hand, one can say in case of a division ring R, $aR = R$ for all $a \in R$. More generally one would like to note as Dheena [5] has introduced the notion of a *Duo-Near-ring* (sub-commutative) which may be considered as analogous to what has been mentioned above. This justifiably motivates us as it is seen in a case of a near-ring [5] to give the notion of the *left* (*right*) *sub-commutativity* in a ring R is as follows. A ring R is *left-sub commutative* if for $a, b \in R$ we have $d \in R$ such that $ab = da$ (*right sub- commutative* if for $a, b \in R$ we have $c \in R$ such that $ab = bc$). The ring R is *sub-commutative*, if for $a, b \in R$ we have $c, d \in R$ such that $ab = bc = da$. A ring R is *-prime sub-commutative* in the sense that for any subset T and prime sub-module S of $R_R, TS = ST$. The number of elements in a set is denoted by # here.

Lemma 3.1: If N_R is a sub-module of M_R then N is a right ideal of M_f.

Proof: Here for $n_1, n_2 \in N$, $n_1 + n_2 \in N$ .And for, $n \in N$ and $m \in M$, $n \bullet_f m = nf(m) \in N$ .Thus N is a right ideal of M_f .However for N - a right ideal of M_f, the additive group $(N, +)$ is a sub module of M_R if f is onto.

Lemma 3.2: For any subset $S(\neq \phi) \subseteq M$, $S \bullet_f M$ is a sub-module of M_R

The following lemma follows immediately from the ring structure of M_f making $\frac{M}{N}$ is a right M_f module.

Lemma 3.3: Let N_R be a sub-module of M_R .Then $\frac{M}{N}$ is a right M_f -module.

Lemma 3.4 (a): Let N_R be a finite sub-module of the module M_R .Then $\frac{N:x}{Annx}$ is a finite R -module. [$N:x = \{r \in R \mid xr \in N\}$].

Proof: We know $Ann_R(x+N) = \{r \in R \mid (x+N)r = N\} = \{r \in R \mid xr + N = N\} = \{r \in R \mid xr \in N\} = N:x$ .Now the map $f: Ann_R(x+N) \to x(Ann_R(x+N))$ defined by $f(\alpha) = x\alpha$ is onto and $= \{\alpha \in Ann_R(x+N) \mid x\alpha = 0\}$ $= \{\alpha \in Ann_R(x+N) \mid \alpha \in Ann_R x\} = Ann_R x$ [since $Ann_R x \subseteq Ann_R(x+N)$].Therefore $\frac{Ann_R(x+N)}{Ann_R x} \cong x(Ann_R(x+N))$ i.e. $\frac{N:x}{Ann_R x} \cong x(N:x)$ .Again $N:x = \{r \in R \mid xr \in N\}$ .Gives $x(N:x) \subseteq N$ and since N is finite, therefore $\frac{N:x}{Ann_R x}$ is finite.

Lemma 3.4(b): Let N_R be a finite sub-module of M_R .Then $\frac{N:x}{r_{M_f}(x)}$ is finite with respect to the ring N_f . [$N_f : x = \{m \in M \mid x \bullet_f m \in N\}$].

Lemma 3.5: Let the module M_R be with an infinite numbers of f *-right-finite* elements. Then M contains an infinite f *-clique*.**[Beck [1]]**

Lemma 3.6: Let I be a left ideal of R . And $(Ann_M I =)N(\subseteq Ker(f))$ is a sub-module of M .Then, $\left(\frac{M}{N}\right)_\psi$ is an *h-ring* for the ψ induced by f where $\psi : \frac{M}{N} \to R$ is such that $\psi(m+N) = f(m)$.

Lemma 3.7: Let $N = Ker(f)$ be a finite sub-module of the module M_R . Then the module M contains an infinite f *-clique* if and only if $\frac{M}{N}$ contains an infinite ψ – clique where ψ is induced by f with $\psi(m+N) = f(m)$.

Proof: Let C be an infinite f *-clique* of M .Then we will show that $\overline{C} = \{c+N \mid c \in C\}$ is a *ψ-clique* of $\frac{M}{N}$ .Now for $c_i + N, c_j + N \in C$,

$(c_i+N)\bullet_\psi (c_j+N)$ $=(c_i+N)\psi(c_j+N)$ $=(c_i+N)f(c_j)$ $=$ $c_i f(c_j)+N$ =0+N $=\bar{0}$ .Similarly $(c_j+N)\bullet_\psi (c_i+N)=\bar{0}\forall i\neq j$ .Thus $\overline{C}=\{c+N \mid c\in C\}$ is a ψ*-clique* of $\frac{M}{N}$ And $(c_i+N)\neq(c_j+N)$, $(i\neq j)$ as $N=Ker(f)$. Conversely, let $\{\overline{c_i}\}_{i=1}^{\infty}$ be an infinite ψ*-clique* of $\frac{M}{N}$.Hence $c_i \bullet_f c_j \in N \forall i\neq j$ .Therefore the set of products $\{c_i \bullet_f c_j\}_{i\neq j}$ is a finite set. Now we can construct an infinite f *-clique* of M as it is done in the Lemma 3.5.

4. Main Results

The main results are here presented as follows.

4.1 Characterizations of Color Module

A module M_R (or $_R M$) is a *zero module* if $M=0$

Theorem 4.1.1: $\chi(M)=1$ if and only if M is a zero module over R .

Note

1. In case of a complete graph $(R, E, 0)$, $\chi(R_0)$ is the order of the ring i.e. $0(R)$
2. If $M(\neq 0)$ is a module over a ring R with unity, then $2\leq\chi(M_f)\leq \#\ M$.
3. For a field F , $\chi(F_f)=2$ for all $f(\neq \hat{0})$ as we have only one such f viz: the identity map.
4. It is to be noted that the chromatic number of a ring R as discussed in [1] appears as a very restricted one. It is not difficult to see that in case of the ring of integer modulo N , when n is other than prime; the *f-chromatic number* varies for different R -homomorphism f .

The problem may be considered as an open one to investigate under what condition one may expect the existence of $\chi(R)$

Here one may note below the independence of the result with what was stated in Beck[1].

Proposition 4.1.2: Let $p_1, p_2, ... p_k, q_1, ..., q_r$ be different primes and $N = p_1^{2n_1}...p_k^{2n_k}.\ q_1^{2m_1+1}......q_r^{2m_r+1}$ and let $f: Z_N \to Z$ be a homomorphism such that $f(\overline{m})$=k, where k is the remainder when m is divided by N .Then $\chi_f(Z_N)$ =clique $(Z_N)_f = p_1^{n_1}...p_k^{n_k}.q_1^{m_1+1}....q_r^{m_r+1}+$ r

Proof: Let $y_0 = p_1^{n_1} \dots p_k^{n_k} . q_1^{m_1+1} \dots\dots q_r^{m_r+1}$ .Then $f(\overline{y_0}) = y_0$ .Consider the ideal $Z_N \bullet_f \overline{\alpha} = \{\overline{z_i} \bullet_f \overline{\alpha} \mid \overline{z_i} \in Z_N\}$ [where $y_0 = f(\overline{y_0}) = f(\overline{\alpha})$, say]. Now $(\overline{z_i} \bullet_f \overline{\alpha}) \bullet_f (\overline{z_j} \bullet_f \overline{\alpha}) = (\overline{z_i} f(\overline{\alpha})) \bullet_f (\overline{z_j} f(\overline{\alpha})) = \overline{z_i} f(\overline{\alpha}) f(\overline{z_j}) f(\overline{\alpha}) = \overline{z_i} f(\overline{z_j}) f(\overline{\alpha}) f(\overline{\alpha}) ==$ $\overline{z_i} f(\overline{z_j}) f(\overline{y_0}) y_0 = \overline{z_i} f(\overline{z_j}) f(\overline{y_0 y_0}) = \overline{z_i} f(\overline{z_j}) f((\overline{y_0})^2) = \overline{z_i} f(\overline{z_j}) f(0) = 0$

Similarly, $(\overline{z_j} \bullet_f \overline{\alpha}) \bullet_f (\overline{z_i} \bullet_f \overline{\alpha})$=0.Thus $Z_N \bullet_f \overline{\alpha}$ is an *f-clique*, whose cardinality is $p_1^{n_1} \dots p_k^{n_k} . q_1^{m_1} \dots\dots q_r^{m_r}$

Then the set $\overline{c} = Z_N \bullet_f \overline{\alpha} \cup \{\overline{y_1} \bullet_f \overline{\alpha}, \dots, \overline{y_n} \bullet_f \overline{\alpha}\}$is an *f-clique* of $(Z_N)_f$ containing $s = p_1^{n_1} \dots p_k^{n_k} . q_1^{m_1} \dots\dots q_r^{m_r} + r$ numbers of elements. Hence clique$(Z_N)_f$ ≥s. In order to show that $\chi(Z_N)_f \le$ s, we attach a distinct color to each of the elements of the *f-clique* $\overline{C}$. Furthermore let $\overline{x_i} \bullet_f \overline{\alpha} = \overline{x_i} f(\overline{\alpha}) = \overline{\frac{N}{p_i^{n_i}}} f(\overline{\alpha})$, where $\overline{x_i} = \overline{\frac{N}{p_i^{n_i}}}, 1 \le i \le k$ .Then $\overline{x_1} \bullet_f \overline{\alpha}, \overline{x_2} \bullet_f \alpha, \dots, \overline{x_k} \bullet_f \overline{\alpha}$ are also elements of $\overline{C}$.[Since $\overline{x_i} f(\overline{\alpha}) \bullet_f \overline{x_j} f(\overline{\alpha})$ =0.And hence have been equipped with a color. Let $c(\overline{y})$ denote the color of an element $\overline{y}$ and color the remaining elements of $(Z_N)_f$ as follows. Pick $\overline{x} \notin Z_N \bullet_f \overline{\alpha}$. If $p_1^{n_1} \cdots p_k^{n_k}$ divides x define, $C(\overline{x}) = C(\overline{y_j} \bullet_f \overline{\alpha})$, where $j = \min\{i \mid q_i^{m_i+1} \nmid x\}$. If $p_1^{n_1} \cdots p_k^{n_k}$ does not divides x define, $C(\overline{x}) = C(\overline{x_j} \bullet_f \overline{\alpha})$, where j = min{i | $p_i^{n_i} \nmid$ x}.It is easily seen that this coloring attaches different colors to *f-adjacent* vertices. Thus $\chi((Z_N)_f) \le s$ Hence $\chi((Z_N)_f) = s$

This is as an example of Goldie module where *f-chromatic number* coincides with f -clique of the module.

Theorem 4.1.3: Let M be a module over a *right- sub commutative ring* R . If the *h-ring* M_f contains a nilpotent element, which is not ***f****-right- finite*, then M_f contains an infinite *f-clique* Beck[1]**.**

Theorem 4.1.4: Let M be an *f-color module* over a *right sub- commutative ring* R and S be any subset of M **.** Then $\frac{M}{r_{M_f}(S)}$ is a *ψ- color module* where ψ is induced by f .

Proof: Let $\{\overline{m_1}, \dots, \overline{m_n}\} \subseteq \frac{M}{r_{M_f}(S)}$ is a *ψ-clique* of $\frac{M}{r_{M_f}(S)}$. Now $\overline{m_i} \bullet_\psi \overline{m_j} = \overline{0}$ gives that

$m_i \bullet_f m_j \in r_{M_f}(S)$. Thus $S \bullet_f (m_i \bullet_f m_j) = 0$ Now $\left\{ \sum_{finite} s_i \bullet_f m_j \mid s_i \in S, m_j \in M \right\}$ is an *f-clique* of $S \bullet_f M$.

Clearly, clique $\frac{M}{r_{M_f}(S)} \leq$ clique $S \bullet_f M$ .Now every *f- clique* of $S \bullet_f M$ is also an *f-clique* of M_f as $S \bullet_f M$ is a sub-ring of M_f .Since M is *f-color module* therefore clique $M_f < \propto$ which gives Clique $S \bullet_f M$ is finite. Thus clique $\frac{M}{r_{M_f}(S)}$ is finite. Thus, $\frac{M}{r_{M_f}(S)}$ is a *ψ-color module.*

4.2 Inheritance of Goldie Character

Theorem 4.2.1: Let M be a module over R .

- **i.** **If** M_f satisfies the a.c.c. on right annihilators of subsets then M also satisfies the a.c.c. on annihilators of subsets of M in R .
- **ii.** If M_f sis finite dimensional then M is also finite dimensional.

Proof: i) We consider an ascending chain $Ann_R(s_1) \subseteq Ann_R(S_2) \subseteq ......$, where $S_i \subseteq M, i = 1,2,....$

We claim for some $t \in Z^+$, $Ann_R(S_t) = Ann_R(S_{t+1}) = ...$ .Now for any ascending chain $Ann_R(s_1) \subseteq Ann_R(S_2) \subseteq ......$, where $S_i \subseteq M, i = 1,2,....$ there corresponds a chain of right annihilators of subsets of M viz: $r_{M_f}(S_1) \subseteq r_{M_f}(S_2) \subseteq ......$ As M_f satisfies the a.c.c. on right annihilators of subsets in M we have $r_{M_f}(S_k) = r_{M_f}(S_{k+1}) = r_{M_f}(S_{k+2}) = ...$ for some $k \in Z^+$ .We claim $Ann_R(S_k) = Ann_R(S_{k+1}) = ...$ Suppose $Ann_R(S_k) \subsetneq Ann_R(S_{k+1})$ .Then there exists some $\alpha \in Ann_R(S_{k+1})$ such that $\alpha \notin Ann_R(S_k)$ which gives $S_k.\alpha \neq 0$ for some $s_k \in S_k$ and $S_{k+1}\alpha = 0$. Thus $s_k \bullet_f m \neq 0$ for some $s_k \in S_k$ and $\sum t_i r_i = m$ [where $Ann(S_k) = \langle f(T) \rangle, T \subset M$] and $S_{k+1}\alpha = 0$. Thus $m \notin r_{M_f}(S)$ but $m \in r_{M_f}(S_{k+1})$,a contradiction. Thus our claim.

ii. Suppose M is not finite dimensional. Now if N_R is a sub-module of M_R then N is a right ideal of M_f [Lemma 3.1]. Thus $N_1 \oplus N_2 \oplus .....$, an infinite direct sum of right ideals of M_f gives rise to an infinite direct sum of right ideals of M_f .

4.3 Prime Modules

Theorem 4.3.1: If M is a prime module over a ring R then $r_R(M)$ is a prime ideal of R .

Proof: Clearly $r_R(M)$ is an ideal. Now let A and B are two ideals of R such that $AB \subseteq r_R(M)$ i.e. $MAB = 0$ .Assume that $A \not\subset r_R(M)$ .Now $MAB = (MA)B \subseteq MB$ .And $MAB = 0$ gives $B \subseteq r_R(MA) = r_R(M)$ .Thus $r_R(M)$ is a prime ideal of R .

The subset $T(S) = \{s \in S \mid sr = 0 \text{ for some non zero } r \in R\}$ of S is the *torsion subset* of S . S is a *torsion set* if $T(S) = S$ and is *torsion-free* if $T(S) = 0$ Goodearl[6, Page 5].

Theorem 4.3.2: Let M_R be *torsion- free* over the *prime sub-commutative* ring R . If M_1 is a prime sub-module of M and R_1 is a prime sub-module of R_R such that $r_R(M_1)$ and $l_R(R_1)$ are different maximal ideals then M_1R_1=0.

Proof: Assume $M_1R_1 \neq 0$. This implies $R_1 \not\subset r_R(M_1)$ and we get $r_R(M_1) : R_1 = r_R(M_1)$ [as $r_R(M_1)$ is a prime ideal].Also $r_R(M_1R_1) = r_R(M_1)$ .Now $x \in l_R(R_1)$ gives $xR_1 = 0$ which gives $x \in r_R(M_1R_1)$

Thus $l_R(R_1) \subseteq r_R(M_1R_1)$ .The torsion-free character of M leads to $l_R(R_1) = r_R(M_1R_1)$ .Thus $l_R(R_1) = r_R(M_1R_1) = r_R(M_1)$, a contradiction. Hence M_1R_1 =0.

Theorem 4.3.3: Let A be a sub -module of M such that $K = Ker(f) \subseteq A$. If $\frac{A}{K} \leq^{prime} \frac{M}{K}$ then $f(A) \leq^{prime} R$.

Proof: Let $\frac{A}{K} \leq^{prime} \frac{M}{K}$ then for any sub- module $\frac{B}{K}$ of $\frac{A}{K}$, $Ann\frac{B}{K} = Ann\frac{A}{K}$ .Here $\frac{B}{K}$ = $\{b + K \mid b \in B\}$. $Ann\frac{B}{K} = \{r \in R \mid (b + K)r = K\})$ $= \{r \in R \mid br \in K\} = \{r \in R \mid f(br) = 0\}$.Let S is a sub -module of $f(A)$.

Thus $AnnS \supseteq Annf(A)$.Consider $A_1 \supseteq A$ such that $f(A_1) = S$. Then $A_1 \leq A$ .Now for $r \in AnnS$, $rs = 0$ for all s∈S gives $f(a)r = 0$ for all $a \in A_1$ .Thus $ar \in Ker(f) = K$ Hence $(a + K)r = K$ i.e. $r \in Ann\frac{A_1}{K} = Ann\frac{A}{K}$ i.e. $(a + K)r = K \ \forall a \in A \Rightarrow ar + K = K$ $\Rightarrow ar \in K$.$\Rightarrow f(ar) = 0 \Rightarrow f(a)r$ for all $a \in A$ gives that $f(A)r = 0$ .Thus $AnnS \subseteq Annf(A)$ i.e. $AnnS = Annf(A)$ .Hence $f(A) \leq^{prime} R$

4.4 Interrelation between Color modules and Goldie modules

The following theorem deals with a module having $\chi(M)\langle\infty$ In this section we explore few ideas to convert a color module into a Goldie module where the so-called *f-clique* are of the type $C = \{x_1, x_2, ..., x_n\}$ with $x_i \bullet_f x_j = 0$ for all $i \langle j$.

Theorem 4.4.1: Let M be a finite dimensional *torsion- free* module over a *right sub-*

commutative ring R. Then M is a right Goldie if it has no infinite *f-clique.*

Proof: Since M_R is a finite dimensional module. Now we consider one strictly ascending chain of right annihilators of subsets of M, say $r_{M_f}(S_1) \subset r_{M_f}(S_2) \subset r_{M_f}(S_3) \subset .....$which is not stationary.

Let $x_i \in r_{M_f}(S_i) \setminus r_{M_f}(S_{i-1}), i = 2,3,... \Rightarrow x_i \in r_{M_f}(S_i)$ and $x_i \notin r_{M_f}(S_{i-1})$.Then $S_i \bullet_f x_i = 0$ and $S_{i-1} \bullet_f x_i \neq 0$ Consider the elements $y_n = s_{n-1} \bullet_f x_n = s_{n-1} f(x_n)$, $n \geq 2$. Then we can show that $y_i \bullet_f y_j = 0$ for $i \neq j$.[since M is torsion free].Put $z_n = y_{i-n}$.Then $\{z_n\}^{\infty}{}_{n=1}$ is an infinite *f-clique*, a contradiction which implies that M is right Goldie.

Theorem 4.4.2: Let M be module over a *right sub- commutative ring* R .If $\chi(M)\langle\infty$ then $T(M)$, the torsion set of M is finite.

We know that a sub ring of a Goldie ring is always Goldie, but the same need not be true in case of its every homomorphic image and this leads to the notion of what is known as a fully Goldie ring [Jategaoenkar [15], Page4], as well as that of a fully Goldie module Chowdhury [4].It is seen that in case of the Z-module Z_6, $Ann(Z_6) = 6Z$ and $M = \{\bar{0}, \bar{2}, \bar{4}\}$ is a sub-module of Z_6 where $AnnM = 3Z$.

Thus Z_6 is not a prime module but the quotient module Z_5 is a prime module. And this motivates us to give the notion is a *fully- prime module* as follows. A *prime module* M is *fully- prime* if every homomorphic image of it is again prime. The Z-module Z_p clearly is an example of a *fully-prime module*. Through out the theorems given below we will consider M is a *fully- prime* over a *-prime sub-commutative* ring R.

Theorem 4.4.3: Let M_R be *torsion -free.* If $\chi(M)\langle\alpha$ then the zero module of M is a finite intersection of prime Sub modules of M .

Proof: Assume M_R be *torsion –free* and $\chi(M)\langle\alpha$ .Then $\chi(M_f)\langle\alpha$ for any R-homomorphism f . Clearly M_f is a reduced. By theorem 4.4.1 it follows that M has the a.c.c on right annihilators of sub-modules of M in R. Let $r_R(S_i),\ i \in I$ be the different maximal members of the family $\{r_R(S\ )| S \neq 0\}$ where S_i's are prime sub-modules of M . We claim I is finite. Suppose not. Now $S_i f(S_j) = 0 \forall i, j \in I$ gives $S_i \bullet_f S_j = 0$ .Thus we will get an infinite *f-clique* of M , a contradiction. Hence claim. Now we claim $\bigcap S_i = \{0\}$ .Let $s(\neq 0) \in \bigcap S_i$ which gives $s \in S_i \forall i$ .Thus $Anns = AnnS_i \neq \{0\}$ $s \in S_i \forall i$ .Thus $Anns = AnnS_i \neq \{0\}$ .Let $r(\neq 0) \in Anns = AnnS_i$.

Then $rs = 0$ which gives that $s = 0$, a contradiction. Thus $\bigcap S_i = \{0\}$

Corollary 4.4.4: Let M be a *torsion-free* module over a *right sub- commutative ring* R and $\chi(M)\langle\alpha$ then the torsion set $T(M)$ is a finite intersection of *prime sub modules* of M .

Theorem 4.4.5: Let M be a *torsion-free* finite dimensional module over a *right sub - commutative ring* R . If $\chi(M)\langle\alpha$ then M is a right Goldie module.

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