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# Analytic Solutions for a Class of Functional Differential Equations

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**Abstract**

We present a theorem proving existence of analytic solutions for a general class of functional differential equations. We illustrate the application of the result by examining a certain special case in detail.

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**Keywords:** Functional differential equations, analytic solution, power series, majorizing series.

## 1. Introduction

We consider the existence of analytic solutions for functional differential equations. Our main theorem is stated in Section 2, in which we construct a majorizing series for a very general class of functional differential equations. Following the proof, we compare our result to theorems in the literature. In Section 3, we consider a special case as an example. The example has a very simple form, but we found it challenging and interesting.

## 2. An Existence Theorem for Analytic Solutions

We now state our main result.

**Theorem 2.1.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be analytic in some neighborhood of $(0, 0, 0)$ and let $g : [0, \tau] \to \mathbb{R}$ with $0 \le g(t) \le t$ for all $t \in [0, \tau]$. Then, there exists an $r > 0$ such that the problem

$$\begin{aligned} x'(t) &= f(t, x(t), x(g(t))) \\ x(0) &= 0 \end{aligned}$$

has an analytic solution for $t \in [0, r]$.

We will make use of the following lemma.

**Lemma 2.2.** **[1]** Let $k, a_1, a_2, \ldots, \in \mathbb{R}$ and $n \in \mathbb{N}$ be given. Assume $\sum_{m=1}^{\infty} a_m t^m$ converges for $|t| < R$. Then, $\left(\sum_{m=1}^{\infty} a_m t^m\right)^n$ converges for $|t| < R$ and there exists a polynomial $p_{n,m}$ of $a_1, \ldots, a_m$ with positive integer coefficients such that

$$\left(\sum_{m=1}^{\infty} a_m t^m\right)^n = \sum_{m=n}^{\infty} p_{n,m} t^m.$$

Proof of theorem (This proof is based on the one in [1]). Since $f$ is analytic, there exist $b_{ijk} \in \mathbb{R}$ such that $f(t, w, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{ijk} t^i w^j y^k$. We will represent $x : \mathbb{R} \to \mathbb{R}$ formally by $\sum_{n=1}^{\infty} a_n t^n$, where the radius of convergence will be determined later. We then have, for $t \in [0, \tau]$,

$$\begin{aligned} &|f(t, x(t), x(g(t)))| \\ &\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left|b_{ijk}\right| t^i \left(\sum_{m=1}^{\infty} |a_m|\, t^m\right)^j \left(\sum_{n=1}^{\infty} |a_n|\, [g(t)]^n\right)^k \\ &\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left|b_{ijk}\right| t^i \left(\sum_{m=1}^{\infty} |a_m|\, t^m\right)^j \left(\sum_{n=1}^{\infty} |a_n|\, t^n\right)^k \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left|b_{ijk}\right| t^i \left(\sum_{m=j}^{\infty} p_{j,m} t^m\right) \left(\sum_{n=k}^{\infty} p_{k,n} t^n\right) \end{aligned}$$

$$= \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\left|b_{ijk}\right| t^i \left(\sum_{m=j+k}^{\infty}\left(\sum_{n=k}^{m-j} p_{j.m-n}p_{k,n}\right)t^m\right)\right)\right)$$

$$= \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\left|b_{ijk}\right| \left(\sum_{m=j+k}^{\infty}\left(\sum_{n=k}^{m-j} p_{j.m-n}p_{k,n}\right)t^{m+i}\right)\right)\right) \tag{(1)}$$

where $p_{i,l}$ is defined as in the lemma for $i \geq 1$, $p_{0,0} = 1$ and $p_{0,l} = 0$ for $l \geq 1$. ■

Consider the right-hand side of (1). If we expand this series, the $t^0$ (constant) coefficient is $|b_{000}|$. There are five expressions which add to be the $t^1$ coefficient: 1) $i = 0, j = 0, k = 0, m = 1, n = 0$, in which we obtain $|b_{000}|\ p_{0,1}p_{0,0}$, 2) $i = 0, j = 0, k = 0, m = 1, n = 1$, in which we obtain $|b_{000}|\ p_{0,0}p_{0,1}$, 3) $i = 0, j = 1, k = 0, m = 1, n = 0$, in which we obtain $|b_{010}|\ p_{1,1}p_{0,0}$, 4) $i = 0, j = 0, k = 1, m = 1, n = 1$, in which we obtain $|b_{001}|\ p_{0,0}p_{1,1}$ and 5) $i = 1, j = 0, k = 0, m = 0, n = 0$, in which we obtain $|b_{100}|\ p_{0,0}p_{0,0}$. It is clear that we can proceed inductively to prove that in general, for $n \in \{0, 1, 2, \ldots\}$ the $t^n$ coefficient is given by

$$\sum_{\substack{i,j,k\\ i+j+k\leq n}} \left|b_{ijk}\right| q_{i,j,k,n}$$

where $q_{i,j,k,n}$ is polynomial of $|a_1|, \ldots, |a_n|$ with positive integer coefficients. Note that these coefficients are not those of $f(t, x(t), x(g(t)))$, but we'll use them for a majorizing series. To define the majorizing series, we let

$$A_1 = |b_{000}|$$

$$A_{n+1} = \frac{1}{n+1} \sum_{\substack{i,j,k\\ i+j+k\leq n}} \left|b_{ijk}\right| q_{i,j,k,n},$$

where $q_{i,j,k,n}$ is the polynomial above, now with variables $A_1, \ldots, A_n$. Since $x(t) = \sum_{n=1}^{\infty} a_n t^n$ formally solves

$$x'(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{ijk}t^i\,[x(t)]^j\,[x(g(t))]^k\,,$$

then

$$y(t) = \sum_{n=1}^{\infty} A_n t^n$$

formally solves the majorizing differential equation

$$y'(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left|b_{ijk}\right| t^i\,[y(t)]^j\,[y(t)]^k\,,$$

so we only have left to show that $\sum_{n=1}^{\infty} A_n t^n$ has a positive radius of convergence.

Let $s > 0$ and let $P_n(s) = \sum_{m=1}^{n} A_m s^m$. Consider the two expressions $P'_{n+1}(s)$ and $\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} \left|b_{ijk}\right| s^i \left[P_n(s)\right]^j \left[P_n(s)\right]^k$. Recalling the way that the $A_m$'s were defined, the coefficient of $s^m$, for $m = 0, 1, \ldots, n$, of $\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} \left|b_{ijk}\right| s^i \left[P_n(s)\right]^j \left[P_n(s)\right]^k$ is $(m+1)A_{m+1}$, which is the same as the $s^m$ coefficient of $P'_{n+1}(s)$, that is,

$$P'_{n+1}(s) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} \left|b_{ijk}\right| s^i \left[P_n(s)\right]^j \left[P_n(s)\right]^k,$$

for $s > 0$, $n = 1, 2, \ldots$. Choose $a$, $b$, and $c$ such that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left|b_{ijk}\right| a^i b^j c^k < \infty.$$

Call that constant $M$, and let $r = \min\left\{\tau, a, \dfrac{b}{M}, \dfrac{c}{M}\right\}$. For all $s \in (0, r]$, we have

$$P_1(s) = |b_{000}|\, s \leq Mr \leq \min\{b, c\}.$$

Inductively, assume for some $n \in \mathbb{N}$ that

$$P_n(s) \leq \min\{b, c\}$$

for all $s \in (0, r]$. Then,

$$P'_{n+1}(s) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} \left|b_{ijk}\right| s^i \left[P_n(s)\right]^j \left[P_n(s)\right]^k$$

$$\leq \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} \left|b_{ijk}\right| a^i b^j c^k \leq M,$$

therefore

$$P_{n+1}(s) \leq Ms \leq Mr \leq \min\{b, c\},$$

so we have shown $P_n(s) \leq \min\{b, c\}$ for all $n = 1, 2, \ldots$ and $s \in (0, r]$. Hence $\sum_{n=1}^{\infty} A_n t^n$ converges on $[0, r]$ and we obtain the desired conclusion that $\sum_{n=1}^{\infty} a_n t^n$ converges on $[0, r]$.

**Remark 2.3.** Because it follows from the hypotheses that $f$ is Lipschitz, the method of successive approximations can be used to verify existence and uniqueness of solutions. The proof above shows that the unique solution must be analytic.

**Remark 2.4.** Note that a lower bound for the radius of convergence is given by $r = \min\left\{\tau, a, \frac{b}{M}, \frac{c}{M}\right\}$, where $a, b, c$ and $M$ are defined in the proof.

**Remark 2.5.** We find it interesting that no regularity assumption needs to be made on $g$. Evidently, the requirement $0 \le g(t) \le t$ "forces" enough regularity to ensure that solutions are analytic.

**Remark 2.6.** There does not appear to be a lot in the literature concerning existence of analytic solutions for functional differential equations. In [2], it is proven that analytic solutions exist for the problem

$$x'(z) = f(x(qz)) + h(z),$$

where $z, q \in \mathbb{C}$, $\|q\| < 1$, $f, h$ are analytic in a neighborhood about the origin and $f(0) = 0$. Our result generalizes the form of the right-hand side of the functional differential equation.

In [3], it is proven that analytic solutions exist for

$$x'(z) = a(z)x(g(z)) + h(z)$$

where $z \in \mathbb{C}$, $a, g, h$ are analytic in a neighborhood of the origin, $g(0) = 0$ and $0 < \left\|g'(0)\right\| < 1$. Again, Our result generalizes the form of the right-hand side of the functional differential equation and allows $g'(0) = 0$ or $g'(0) = 1$.

In [4], existence of analytic solutions is proven for the problem

$$f(z)x'(z) = \sum_{j=1}^{m} p_j f(x(q_j z))$$

where $z \in \mathbb{C}$, $f$ is analytic in a neighborhood of the origin, $f(0) = 0$ and $f'(0) \neq 0$, $\sum_{j=1}^{m} \left|p_j\right| \le 1$ and $\left|q_j\right| \le 1$ for all $j = 1, 2, \ldots, m$. Our functional differential equation generalizes the form of this differential equation.

More discussion, results and examples can be found in [5].

**Corollary 2.7.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be analytic in some neighborhood of $(0, 1, 1)$ and let $k \in (0, 1]$. Then, there exists an $r > 0$ such that

$$\begin{aligned} x'(t) &= f(t, x(t), x(kt)) \\ x(0) &= 1 \end{aligned}$$

has an analytic solution for $t \in [0, r]$.

## 3. A Special Case

To illustrate our results, we will investigate in detail the special case (SC)

$$x'(t) = x(kt)$$

$$x(0) = 1$$

for $0 < k \leq 1$. From our corollary, we can quickly conclude (SC) has a unique solution that is analytic on $[0, \infty)$, but after an exhaustive search, there appears to be no closed-form solution for $0 < k < 1$, so the power series approach is needed. Before we consider this, we will first note a couple of properties that can be verified without the use of power series.

**Proposition 3.1.** $x$ is increasing. This is easy to see, because $x(0) = 1$ implies that $x$ is positive for small $t$ and hence $x'(t) = x(kt) > 0$.

**Proposition 3.2.** $x$ is concave up. Verification is similar to Proposition 3.1. It can be easily shown that the power series solution to (SC) is given by

$$x(t) = \sum_{n=0}^{\infty} \frac{k^{(n-1)n/2}}{n!} t^n.$$

One can verify directly that it has an infinite radius of convergence.

**Proposition 3.3.** Let $u_i$ represent the solution to (SC) for $k = k_i$, $i = 1, 2$, where $k_1 \leq k_2$. Then, $u_1(t) \leq u_2(t)$ for all $t \geq 0$.

This is easy to see by comparing the power series for $u_1$ and $u_2$ term-by-term.

**Proposition 3.4.** $x(t) \leq e^t$ for all $k \in (0, 1]$. This follows quickly from Proposition 3 by letting $k_2 = 1$.

**Proposition 3.5.** For each $k \in (0, 1]$, $x$ is Lipschitz with constant $e^k$.

*Proof.* Note that

$$x'(t) = x(kt) \leq e^{kt} \tag{(2)}$$

by Proposition 3.4, and hence for any $s, t \in [0, 1]$, we have

$$|x(t) - x(s)| = \left| \int_s^t x'(r) dr \right| \leq \left| \int_s^t e^{kr} dr \right| \leq e^k \, |t - s| \, .$$

■

In what follows let $u_k$ denote the solution to (SC) for each $k \in (0, 1)$.

**Proposition 3.6.** $\lim\limits_{k \to 1} u_k(t) = e^t$ uniformly on $[0, 1]$.

*Proof.* Using long division and the fact that $k < 1$, we have:

$$\left|e^t - u_k(t)\right| = \left|\sum_{n=0}^{\infty}(1-k^{\frac{(n-1)n}{2}})\frac{t^n}{n!}\right| = (1-k)\left|\sum_{n=0}^{\infty}(1+k+k^2+\cdots+k^{\frac{n(n-1)}{2}-1})\frac{t^n}{n!}\right| \leq$$

$$(1-k)\sum_{n=0}^{\infty}\frac{(n-1)n}{2}\frac{t^n}{n!} = (1-k)\sum_{n=0}^{\infty}\frac{t^n}{2(n-2)!}.$$

Note that this last series describes an entire function since the ratio test yields $\lim_{n\to\infty}\frac{\frac{1}{2(n-1)!}}{\frac{1}{2(n-2)!}} = \lim_{n\to\infty}\frac{1}{n-1} = 0$. This series is, therefore, bounded in modulus on [0,1] by some positive value $M$. Thus, we have $\left|e^t - u_k(t)\right| \leq (1-k)M \to 0$ as $k \to 1$. This limit is clearly uniform in $t$. ■

**Proposition 3.7.** An error bound in using the first $n$ terms of the power series to approximate the true solution on [0, 1] is given by

$$\frac{k^{\frac{n^2+n-2}{2}}e^{k^{n+2}}}{(n+1)!}.$$

*Proof.* To verify this, we need only bound the well-known remainder term $R_n(t) = \frac{x^{(n+1)}(\varsigma(t))\,t^{n+1}}{(n+1)!}$, where $0 \leq \varsigma(t) \leq t$. First, we claim $x^{(i)}(t) = k^{(i-1)i/2}x(k^i t)$ for all $i = 1, 2, \ldots, n+1$. We note that for $i = 1$, this is simply $x'(t) = x(kt)$. Proceeding inductively, assume

$$x^{(j)}(t) = k^{(j-1)j/2}x(k^j t)$$

for some $j = 1, 2, \ldots, n$. Then,

$$x^{(j+1)}(t) = \frac{d}{dt}\left[k^{(j-1)j/2}x(k^j t)\right] = k^{\frac{j^2}{2}-\frac{j}{2}}k^j x'(k^j t) = k^{\frac{j(j+1)}{2}}x(k^{j+1}t)$$

as desired. We then have, also applying the inequality $x(t) \leq \frac{e^{kt}}{k} + 1 - \frac{1}{k}$ (see (3) below)

$$\begin{aligned}|R_n(t)| \leq \left|\frac{x^{(n+1)}(\varsigma(t))\,t^{n+1}}{(n+1)!}\right| &= \frac{k^{\frac{n(n+1)}{2}}x(k^{n+1}\varsigma(t))t^{n+1}}{(n+1)!}\\ &\leq \frac{k^{\frac{n(n+1)}{2}}\left(\frac{e^{k\left(k^{n+1}\varsigma(t)\right)}}{k} + 1 - \frac{1}{k}\right)t^{n+1}}{(n+1)!}\\ &\leq \frac{k^{\frac{n^2+n-2}{2}}e^{k^{n+2}}}{(n+1)!}.\end{aligned}$$

Table 1: Approximations Using Truncated Power Series.

| $t$ | $x(t)$ |
|---|---|
| 0.1 | 1.102520898 |
| 0.2 | 1.210167710 |
| 0.3 | 1.323067793 |
| 0.4 | 1.441350083 |
| 0.5 | 1.565145111 |
| 0.6 | 1.694585009 |
| 0.7 | 1.829803521 |
| 0.8 | 1.970936011 |
| 0.9 | 2.118119476 |
| 1.0 | 2.271492555 |

Using the first 100 terms of the power series, we approximated the solution for the case $k = \frac{1}{2}$:

While it can be easily shown that (SC) has no solution of the form $x = ab^t$, the data above leads us to believe that the solution is exponential-like. In fact, when one does a least squares curve fit of the form $x = ab^t$ to the data above, one obtains $x = 1.03549517(2.227166859^t)$ with an $r$ of 0.998969153! Applying Proposition 7, we get an error bound of $\frac{\left(\frac{1}{2}\right)^{\frac{100^2+100-2}{2}} e^{\left(\frac{1}{2}\right)^{100+2}}}{(100+1)!} < \frac{\left(\frac{1}{2}\right)^{50} \cdot 3}{1} \leq 3 \times 10^{-15}$, so ignoring round-off errors, Table 1 is accurate to all nine decimal places.

We can also derive a closed form upper bound for the solution of (SC): Integrating (2) from 0 to $t$, we have

$$\int_0^t x'(s)ds \leq \int_0^t e^{ks} ds$$

and hence

$$x(t) \leq \frac{e^{kt}}{k} + 1 - \frac{1}{k}. \tag{(3)}$$

For example, for $k = \frac{1}{2}$, we have $x(t) \leq 2e^{\frac{t}{2}} - 1$. We wondered if this could be used as an approximate closed form solution in addition to serving as an upper bound. To get an idea of this, we proceeded as follows. Finding a least squares fit to the data in Table 1 of the form $x = ae^{bt} - 1$, we obtain $x = 2.004319707(1.63481654^t) \approx 2.004319707e^{.4915305901t}$ with an $r$ of .9999800998, so the choice of the constants $a = 2$ and $b = .5$ look fairly good. Also, by constructing a table for the approximation $x = 2e^{t/2} - 1$ as we did for the power series approximation above we obtain rounded to nine decimal places. The difference in the two approximations for $x(1)$ is about 1.1%.

Table 2: Approximations Using $x = 2e^{t/2} - 1$.

| $t$ | $x(t)$ |
|---|---|
| 0.1 | 1.102542913 |
| 0.2 | 1.210341836 |
| 0.3 | 1.323668485 |
| 0.4 | 1.442805516 |
| 0.5 | 1.568050833 |
| 0.6 | 1.699717615 |
| 0.7 | 1.838135097 |
| 0.8 | 1.983649395 |
| 0.9 | 2.136624371 |
| 1.0 | 2.297442541 |

So $x = 2e^{t/2} - 1$ provides not only a simple upper bound for the solution, but even a fairly decent approximate solution.

Finally, we make note of the following curious fact. Consider (SC) in the case in which $k$ is negative with $-1 \leq k < 0$. Let $x : [0, \infty) \to \mathbb{R}$ be *any* differentiable function with $x(0) = 1$. Extend $x$ to all of $(-\infty, \infty)$ by letting $x(t) = x'\left(\frac{t}{k}\right)$ on $(-\infty, 0)$. Then, for $t \geq 0$, $x(kt) = x'\left(\frac{kt}{k}\right) = x'(t)$ and hence $x$ is a solution to (SC).So, the situation is much different for negative $k$ from the case in which $k$ is positive. ■

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