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Congruence Lattice of Algebraic Standard Lattice

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Abstract

In this paper, the congruence lattice of algebraic standard lattice using distributive, dually distributive and standard algebraic lattices are established.

Keywords: Algebraic lattice, algebraic distributive lattice, algebraic dually distributive lattice algebraic standard lattice.

Introduction

The concept of algebraic lattice was already introduced by G. Grätzer [4], G. Birkhoff, O. Frink [1], Steven Roman [5] and Funayama and Nakayama [3]. Peter Crawley and Dilworth [2] explained some concepts on Algebraic theory of lattices. Algebraic lattices originated with Komatu and Nachdin in the 1940's and Büchi in the early 1950's.

In this paper, we explain the relations between congruence lattice and algebraic distributive lattice, algebraic dually distributive lattice, algebraic standard lattice. If L is an arbitrary lattice then congruence lattice $Con\,L$ is an algebraic lattice.

Definition: 1

A complete lattice L is an algebraic lattice or compactly generated, if it is complete and $K(L)$ is join dense in L , $x = \bigvee\left(\downarrow x \cap K(L)\right)$, for every $x \in L$, Since $K(L)$ is a compact element.

Example: 1

Every finite lattice is algebraic.

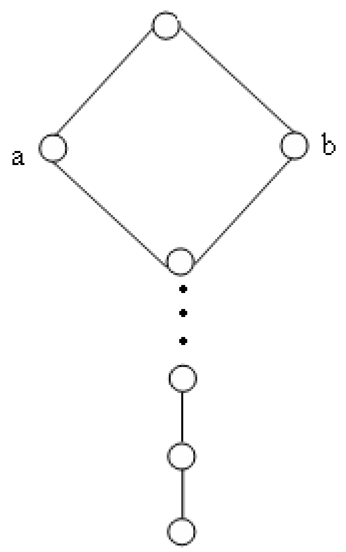
For, let L be any finite lattice. Let a , $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$ (since L is finite). Let $K \subseteq L \Rightarrow K = \{a,b\}$. Therefore $\bigvee K \subseteq L$ and $\Lambda K \subseteq L$. Hence K is complete. In general, every element of a complete lattice K is compact and all elements of finite lattice are compact. That is $K = K(L)$. Therefore K is compact. Hence L is algebraic.

Example: 2

A complete lattice need not be an algebraic lattice.

For, let K be a complete lattice

Let K denote the interval $[0,1]$ in the real number with the usual order. Then $K(L) = \{0\}$. Therefore K is not algebraic.



Definition: 2

An algebraic lattice $A\lg(L)$ is called an algebraic distributive lattice if it is distributive, that is $A \vee (x \wedge y) = (A \vee x) \wedge (A \vee y)$, for all x, y in $A\lg(L)$ and "A" is an distributive element of $A\lg(L)$

Definition: 3

An algebraic lattice $A\lg(L)$ is called an algebraic dually distributive lattice if it is dually distributive, that is $A \wedge (x \vee y) = (A \wedge x) \vee (A \wedge y)$, for all x, y in $A\lg(L)$ and "A" is an dually distributive element of $A\lg(L)$.

Definition: 4

An algebraic lattice $A\lg(L)$ is called an algebraic standard lattice if it is standard, that is $x \wedge (A \vee y) = (x \wedge A) \vee (x \wedge y)$, for all x, y in $A\lg(L)$ and "A" is an standard element of $A\lg(L)$

Example: 3

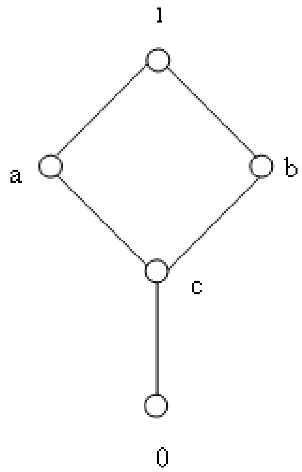
The lattice $C_1 + B_2$ is algebraic distributive lattice.

For, consider a lattice $C_1 + B_2$

Let $a, b, c \in L$ such that $a \wedge b = 0$, $a \wedge c = 0$ and $b \wedge c = 0$

And $a \vee b = 1$, $a \vee c = a$ and $b \vee c = b$ .Let $K = \{a, b, c\}$ be the subset of L. The least upper bound and great lower bound are contained in L. That is $\bigvee K$ and $\bigwedge K$ are in L.

Then K is complete. Every finite set is compact, then K is compact. Hence K is algebraic. Let K be an algebraic lattice and $(a \vee b) \wedge (a \vee c) = 1 \wedge (a) = a$ then $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, for all $a, b, c \in K$ .Therefore L is an algebraic distributive lattice.



Theorem: 1

Let L be an arbitrary lattice then congruence lattice $Con\, L$ is an algebraic lattice.

Theorem: 2

Let L be an arbitrary lattice then $Con\, L$ is an algebraic distributive lattice.

Theorem: 3

Let L be an arbitrary lattice then $Con\, L$ is an algebraic dually distributive lattice.

Theorem: 4

Let L be an arbitrary lattice, then $Con\, L$ is an algebraic standard lattice.

Proof:

We have $Con\, L$ is an algebraic lattice. Enough to prove, $\Theta(L)$ is standard

Before taking up the standard of $\Theta(L)$, we need the following observation.

If $a,b \in L$, $\theta \in \Theta(L)$ and $a \wedge b \le c$, $a \vee b \ge c$

$\Rightarrow$ $a \vee b \ge c \ge a \wedge b$ then $a \equiv b(\theta)$

Let $a \equiv b(\theta)$ then $a \vee b \equiv b \vee b(\theta)$, (by substitution property)

$\Rightarrow$ $a \vee b \equiv b(\theta)$, (by idempotent law)

$\Rightarrow$ $b \equiv a \vee b(\theta)$, ($\because$ θ is symmetric)

$\Rightarrow$ $a \wedge b \equiv a \vee b(\theta)$, ($\because$ θ is transitive)

and $a \wedge b \equiv a \vee b(\theta)$ implies that
$c = (a \vee b) \wedge c \equiv [(a \wedge b) \wedge c](\theta) = a \wedge b$ thus $c = a \wedge b$

Therefore two elements a,b belong to the same θ - class if and only if every element in the quotient sublattice $a \vee b / a \wedge b$ belongs to the same θ - class.

Now, to prove $\phi \wedge (\theta \vee \psi) = (\phi \wedge \theta) \vee (\phi \wedge \psi)$, for all ϕ, ψ in $Con\,L$

Let $\theta \in \Theta(L)$ and ϕ be a subset of $\Theta(L)$

If $\sigma = \vee\theta$ and $\tau = \bigvee_{\psi \in \theta} (\phi \wedge \psi)$ then $\phi \wedge \sigma \ge \tau$

Claim: $\phi \wedge \sigma \ge \tau$
Suppose $a \equiv b(\phi \wedge \sigma)$ then $a \equiv b(\phi)$ and $a \equiv b(\sigma)$ (1)

Take $a \equiv b(\phi)$, hence there exists a sequence $a = a_0, a_1, a_2 \ldots. a_n = b$ of elements in L and congruence relations $\psi_1, \psi_2 \ldots \psi_n \in \theta$ such that $a_{j-1} \equiv a_j(\phi_j)$, for each $j = 1,2 \ldots n$.

Set $c_j = (a \vee b) \wedge (a_j \vee a_{j+1} \vee \ldots. \vee a_n)$, for all $j = 1,2 \ldots., n$

Then $a \vee b = c_0 \ge c_1 \ge c_2 \ge \ldots \ge c_n = b$ and $a \vee b \equiv b(\phi)$. Hence $c_{j-1} \equiv c_j(\phi)$

But $c_{j-1} \equiv [(a \vee b) \wedge (a_j \vee a_{j+1} \vee \ldots. \vee a_n)](\psi_j) = c_j$, since $a_{j-1} \equiv a_j(\psi_j)$ and $c_{j-1} \equiv c_j(\phi \wedge \psi_j)$, for each $j = 1,2 \ldots., n$. Thus $a \vee b \equiv b(\tau)$, Take $a \equiv b(\sigma)$, hence there exists a sequence $a = a_0, a_1, a_2 ... a_n = b$ of elements in L and congruence relations $\theta_1, \theta_2 ... \theta_n \in \phi$ such that $a_{j-1} \equiv a_j(\phi_j)$, for each $j = 1,2 ... n$

Set $c_j = (a \vee b) \wedge (a_j \vee a_{j+1} \vee .... \vee a_n)$, for all $j = 1,2 ...., n$

Then we have $a \vee b = c_0 \geq c_1 \geq c_2 \geq ... \geq c_n = b$ and as $a \vee b \equiv b(\phi)$.

Hence

$$c_{j-1} \equiv c_j(\phi)$$

But $c_{j-1} \equiv \left[(a \vee b) \wedge (a_j \vee a_{j+1} \vee ... \vee a_n)\right](\theta_j) = c_j$, since $a_{j-1} \equiv a_j(\theta_j)$ and $c_{j-1} \equiv c_j(\phi_j \wedge \theta)$, for each $j = 1,2 ... n$. Thus $a \vee b \equiv a(\tau)$ so that $a \equiv b(\tau)$ (2)

From (1) and (2), $\phi \wedge \sigma \geq \tau$ (3)

Let $a \equiv b(\tau)$ (4)

Claim: $\tau \leq \phi \wedge \sigma$

Suppose $a \equiv b(\tau)$ and hence there exists a sequence $a = a_0, a_1, a_2 ...., a_n = b$ of elements in L and congruence relations $\theta_1, \theta_2 ...., \theta_n \in \theta$ or $\psi_1, \psi_2 ...., \psi_n \in \phi$ such that $a_{j-1} \equiv a_j(\phi_j \wedge (\theta \vee \psi_j))$, for each $j = 1,2 ...., n$.

Set $c_j = (a \vee b) \wedge (a_j \vee a_{j+1} \vee .... \vee a_n)$, for $j = 1,2 ...., n$

Then we have $a \vee b = c_0 \geq c_1 \geq c_2 \geq ... \geq c_n = b$ and $a \vee b \equiv b(\phi) \Rightarrow c_{j-1} \equiv c_j(\phi)$

But $c_{j-1} \equiv \left[(a \vee b) \wedge (a_j \vee a_{j+1} \vee .... \vee a_n)\right](\theta \vee \psi_j) = c_j$

Since $a_{j-1} \equiv a_j(\phi_j \wedge (\theta \vee \psi_j))$ and $c_{j-1} \equiv c_j(\phi_j \wedge (\theta \vee \psi_j))$, for each $j = 1,2 ...., n$

Thus $a \vee b \equiv b(\phi \wedge \sigma)$ and $a \vee b \equiv a(\phi \wedge \sigma)$. Hence $a \equiv b(\phi \wedge \sigma)$ (5)

From (4) and (5), $\tau \leq \phi \wedge \sigma$ 6)

From (3) and (6), $\tau = \phi \wedge \sigma$

Hence $\phi \wedge (\theta \vee \psi) = (\phi \wedge \theta) \vee (\phi \wedge \psi)$, for all ϕ, ψ in $Con L$.

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