International Journal: Mathematical Manuscripts.
ISSN 0974-2883 Volume 5, Number 1 (2012), pp. 17-22

http://www.ripublication.com/ijmm.htm

Equivalent Conditions of Distributive Congruence Lattice

M. Jeyakumar

Lecturer, Department of Mathematics,
Alagappa University Evening College, Rameswaram,
Tamilnadu-623 526, India

Abstract

In this paper, the equivalent conditions of distributive congruence lattice using distributive, dually distributive, congruence homomorphism are established.

Keywords: Congruence lattice, Distributive congruence, Dually distributive congruence, Congruence homomorphism

Introduction
The concept of distributive congruence lattice was discussed by G. Grätzer and E.T. Schmidt in 1958. The distributivity of congruence lattices of lattices has a number of important consequences. B. Johnson [1967] discovered that many of these results hold for arbitrary universal algebras with distributive congruence lattices. A few definitions and results are listed that the equivalent conditions of the distributive congruence lattice using distributive, dually distributive and detail are proved.

Definition: 2.1
A congruence lattice $Con\,L$ of a lattice is distributive if and only if $\lambda \vee (\phi \wedge \psi) = (\lambda \vee \phi) \wedge (\lambda \vee \psi)$, for all ϕ, ψ in $Con\,L$ and ‘ λ ’ is an element of congruence distributive.

Definition: 2.2
A congruence lattice $Con\,L$ of a lattice is dually distributive if and only if $\lambda \wedge (\phi \vee \psi) = (\lambda \wedge \phi) \vee (\lambda \wedge \psi)$, for all ϕ, ψ in $Con\,L$ and ‘ λ ’ is an element of dually congruence distributive.

Definition: 2.3
A congruence homomorphism φ of the congruence lattice $Con\,L$, for all α, β in $Con\,L$ is the map $\varphi : Con\,L \to Con\,L_1$ is defined by

$$(\alpha \wedge \beta)\varphi = \alpha\varphi \wedge \beta\varphi$$
$$(\alpha \vee \beta)\varphi = \alpha\varphi \vee \beta\varphi.$$

Theorem: 2.1
Let $Con\,L$ be a congruence lattice in a lattice L and D denote the set of all distributive congruence element of $Con\,L$. If α, β in D then $\alpha \vee \beta$ in D.

Proof
Let D be the set of all distributive congruence element of $Con\,L$

Let α, β in D and let ϕ, ψ in $Con\,L$

$$\begin{aligned}(\alpha \vee \beta) \vee (\phi \wedge \psi) &= \alpha \vee \beta \vee (\phi \wedge \psi)\\ &= \alpha \vee [\beta \vee (\phi \wedge \psi)]\\ &= \alpha \vee [(\beta \vee \phi) \wedge (\beta \vee \psi)], \text{(since "}\beta\text{" is distributive)}\\ &= [\alpha \vee (\beta \vee \phi)] \wedge [\alpha \vee (\beta \vee \psi)], \text{(since "}\alpha\text{" is distributive)}\\ &= [(\alpha \vee \beta) \vee \phi] \wedge [(\alpha \vee \beta) \vee \psi]\end{aligned}$$

Therefore $\alpha \vee \beta$ in D

Theorem: 2.2
Let $Con\,L$ be a congruence lattice in a lattice L and D denote the set of all dually distributive congruence element of $Con\,L$. If α, β in D then $\alpha \wedge \beta$ in D.

Proof
Let D be the set of all dually distributive congruence element of $Con\,L$

Let α, β in D and let ϕ, ψ in $Con\,L$

$$\begin{aligned}(\alpha \wedge \beta) \wedge (\phi \vee \psi) &= \alpha \wedge \beta \wedge (\phi \vee \psi)\\ &= \alpha \wedge [\beta \wedge (\phi \vee \psi)]\\ &= \alpha \wedge [(\beta \wedge \phi) \vee (\beta \wedge \psi)], \text{(since "}\beta\text{" is dually distributive)}\\ &= [\alpha \wedge (\beta \wedge \phi)] \vee [\alpha \wedge (\beta \wedge \psi)], \text{(since "}\alpha\text{" is dually distributive)}\\ &= [(\alpha \wedge \beta) \wedge \phi] \vee [(\alpha \wedge \beta) \wedge \psi]\end{aligned}$$

Therefore $\alpha \wedge \beta$ in D

Theorem: 2.3
Let $Con\,L$ be congruence lattice of a lattice L, then the following conditions on $Con\,L$ are equivalent:

1. $Con\,L$ is a distributive congruence lattice.
2. The map $\varphi : \phi \to \lambda \vee \phi$, for all ϕ in $Con\,L$ is a homomorphism $Con\,L$ onto $[\lambda)$ for any element λ of $Con\,L$.
3. The binary relation Θ_{ConL} on $Con\,L$ defined by $\phi \equiv \psi\ (\Theta_{ConL})$ if and only if $\phi \vee \lambda = \psi \vee \lambda$, for some λ in $Con\,L$, where ϕ, ψ in $Con\,L$ such that $\phi \leq \psi$ is a congruence relation.

Proof
Let $Con\,L$ be congruence lattice in a lattice L.
Suppose (i) is true, that is, $Con\,L$ is a distributive congruence lattice.

To prove (i): $\Rightarrow$(ii)
Let λ be any element of $Con\,L$.
Define the map $\varphi : Con\,L \to [\lambda)$ by $\varphi(\phi) = \lambda \vee \phi$, for all ϕ in $Con\,L$

To prove: φ is homomorphism
Claim: φ is one to one
Let φ, ψ in $Con\,L$. Then $\varphi(\phi) = \varphi(\psi)$ $\Rightarrow \lambda \vee \phi = \lambda \vee \psi$ $\Rightarrow \phi = \psi$
Therefore φ is one to one

Claim: φ is onto
Let λ be distributive then there exists μ in $Con\,L$ such that $\mu \geq \lambda$.
Therefore $\varphi(\mu) = \lambda \vee \mu = \mu$, since $\mu \geq \lambda$ Hence φ is onto

Claim: φ is homomorphism
Let φ, ψ in $Con\,L$
Then $\varphi(\phi) \vee \varphi(\psi) = (\lambda \vee \phi) \vee (\lambda \vee \psi) = \lambda \vee (\phi \vee \psi) = \varphi(\phi \vee \psi)$
And $\varphi(\phi) \wedge \varphi(\psi) = (\lambda \vee \phi) \wedge (\lambda \vee \psi) = \lambda \vee (\phi \wedge \psi) = \varphi(\phi \wedge \psi)$.

Hence φ is homomorphism

To prove (ii) $\Rightarrow$(iii)
Suppose (ii) is true, that is, the map $\varphi : \phi \to \lambda \vee \phi$, for all ϕ in $Con\,L$ is a homomorphism $Con\,L$ onto $[\lambda)$

To prove: (iii)

That is to prove: The binary relation $\Theta_{Con\,L}$ on $Con\,L$ defined by $\phi \equiv \psi\ (\Theta_{Con\,L})$ if and only if $\phi \vee \lambda = \psi \vee \lambda$, for some λ in $Con\,L$ is congruence relation.

Let λ be any element of $Con\,L$.

Define the map $\varphi : Con\,L \to [\lambda\,)$ by $\varphi(\phi) = \lambda \vee \phi$, for all ϕ in $Con\,L$

To prove: $\Theta_{Con\,L}$ is reflexive.

Since $\varphi(\phi) = \varphi(\psi)$, for all ϕ, ψ in $Con\,L$

By definition of $\Theta_{Con\,L}$, $\phi \equiv \phi(\Theta_{Con\,L})$ if and only if $\varphi(\phi) = \varphi(\phi)$, for all ϕ in $Con\,L$

$\Rightarrow\ \phi \vee \lambda = \phi \vee \lambda$ .Thus $\Theta_{Con\,L}$ is reflexive.

To prove: $\Theta_{Con\,L}$ is symmetric.

Suppose $\phi \equiv \psi(\Theta_{Con\,L})$, for all ϕ, ψ in $Con\,L$ then $\varphi(\phi) = \varphi(\psi)$

$\Rightarrow\ \phi \vee \lambda = \psi \vee \lambda\ \Rightarrow\ \psi \vee \lambda = \phi \vee \lambda\ \Rightarrow\ \varphi(\psi) = \varphi(\phi)\ \Rightarrow\ \psi \equiv \phi(\Theta_{Con\,L})$

Thus $\Theta_{Con\,L}$ is symmetric.

To prove: $\Theta_{Con\,L}$ is transitive.

Suppose $\phi \le \psi \le \eta$, $\phi \equiv \psi(\Theta_{Con\,L})$ and, $\phi \equiv \eta(\Theta_{ConL})$ for all ϕ, ψ, η in $Con\,L$ then $\varphi(\phi) = \varphi(\psi)$ and $\varphi(\psi) = \varphi(\eta) \Rightarrow\ \phi \vee \lambda = \psi \vee \lambda$ and $\psi \vee \lambda = \eta \vee \lambda$

$\Rightarrow\ \phi \vee \lambda = \eta \vee \lambda\ \Rightarrow\ \varphi(\phi) = \varphi(\eta) \Rightarrow \phi \equiv \eta(\ \Theta_{Con\,L})$

Thus $\Theta_{Con\,L}$ is transitive.

To prove: Substitution property

Suppose $\phi \equiv \phi_1(\Theta_{Con\,L})$ and $\psi \equiv \psi_1(\Theta_{Con\,L})$, for all ϕ, ψ in $Con\,L$

Then $\phi \vee \lambda = \phi_1 \vee \lambda$ and $\psi \vee \lambda = \psi_1 \vee \lambda \Rightarrow \varphi(\phi) = \varphi(\phi_1)$ and $\varphi(\psi) = \varphi(\psi_1)$

Now $\varphi(\phi \vee \psi) = \varphi(\phi) \vee \varphi(\psi)$, (since "$\varphi$" is homomorphism)

$= (\phi \vee \lambda) \vee (\psi \vee \lambda) = (\phi_1 \vee \lambda) \vee (\psi_1 \vee \lambda) = \varphi(\phi_1) \vee \varphi(\psi_1) = \varphi(\phi_1 \vee \psi_1)$

Therefore $\varphi(\phi \vee \psi) = \varphi(\phi_1 \vee \psi_1)$

Hence $(\phi \vee \psi) \equiv (\phi_1 \vee \psi_1)(\Theta_{Con\,L})$

And $\varphi(\phi \wedge \psi) = \varphi(\phi) \wedge \varphi(\psi)$, (since "$\varphi$" is homomorphism)

$= (\phi \vee \lambda) \wedge (\psi \vee \lambda) = (\phi_1 \vee \lambda) \wedge (\psi_1 \vee \lambda) = \varphi(\phi_1 \wedge \psi_1)$

Therefore $\varphi(\phi \wedge \psi) = \varphi(\phi_1 \wedge \psi_1)$

Hence $(\phi \wedge \psi) \equiv (\phi_1 \wedge \psi_1)(\Theta_{Con\,L})$

Therefore $\Theta_{Con\,L}$ satisfies the substitution property.

Hence $\Theta_{Con\,L}$ is a congruence relation.

To prove (iii) $\Rightarrow$ (i)

Suppose (iii) is true, that is, The binary relation Θ_{ConL} on $Con\,L$ defined by $\phi \equiv \psi\ (\Theta_{ConL})$ if and only if $\phi \vee \lambda = \psi \vee \lambda$, for some λ in $Con\,L$ is congruence relation.

To prove: (i)

That is to prove: $Con\,L$ is distributive congruence lattice

Let ϕ, ψ in $Con\,L$ and λ be a distributive element of $Con\,L$

Enough to prove: $\lambda \vee (\phi \wedge \psi) = (\lambda \vee \phi) \wedge (\lambda \vee \psi)$, where λ is a distributive element .

Here $\Theta_{Con\,L}$ is defined by $\phi \equiv \psi\ (\Theta_{Con\,L})$

$\Rightarrow (\phi \wedge \psi) \vee \lambda = (\phi \vee \psi)$, for some λ in $Con\,L$

$\Rightarrow (\phi \wedge \psi) \vee \lambda_1 = (\phi \vee \psi)$, for some $\lambda_1 \leq \lambda$

$\Rightarrow (\phi \wedge \psi) \vee \lambda = (\phi \vee \psi)$, in particular $\lambda_1 = \lambda$

$\Rightarrow [\phi \wedge (\lambda \vee \phi)] \vee \lambda = \phi \vee (\lambda \vee \phi)$, where $\psi = \lambda \vee \phi$

$\Rightarrow (\lambda \vee \phi) \vee \lambda = (\lambda \vee \phi)$,

Therefore $\phi \equiv (\lambda \vee \phi)\ (\Theta_{Con\,L})$

And $\phi \equiv \psi\ (\Theta_{Con\,L}) \Rightarrow (\phi \wedge \psi) \vee \lambda = (\phi \vee \psi)$, for some λ in $Con\,L$

$\Rightarrow (\phi \wedge \psi) \vee \lambda_1 = (\phi \vee \psi)$, for some $\lambda_1 \leq \lambda$

$\Rightarrow (\phi \wedge \psi) \vee \lambda = (\phi \vee \psi)$, in particular $\lambda_1 = \lambda$

$\Rightarrow [(\lambda \vee \psi) \wedge \psi] \vee \lambda = (\lambda \vee \psi) \vee \psi$, where $\phi = \lambda \vee \psi$

$\Rightarrow (\lambda \vee \psi) \vee \lambda = (\lambda \vee \psi)$,

Therefore $\psi \equiv (\lambda \vee \psi)\ (\Theta_{Con\,L})$

Therefore $\phi \wedge \psi \equiv (\lambda \vee \phi) \wedge (\lambda \vee \psi)\ (\Theta_{Con\,L})$

By the definition of $\Theta_{Con\,L}$,

$$\lambda \vee \left[(\phi \wedge \psi) \wedge (\lambda \vee \phi) \wedge (\lambda \vee \psi)\right] = (\phi \wedge \psi) \vee \left[(\lambda \vee \phi) \wedge (\lambda \vee \psi)\right]$$

$$\Rightarrow \lambda \vee (\phi \wedge \psi) = \lambda \vee \left[(\lambda \vee \phi) \wedge (\lambda \vee \psi)\right]$$

$$\Rightarrow \lambda \vee (\phi \wedge \psi) = (\lambda \vee \phi) \wedge (\lambda \vee \psi)$$

$\Rightarrow \lambda$ is distributive congruence lattice.

Hence $Con\,L$ is distributive congruence lattice.

Reference

[1] Birkhoff. G, Lattice Theory, rev. ed. Amer. Math. Soc. New York, 1948.

[2] Funayama. N and Nakayama .T, On the distributivity of a lattice of lattice congruences, Proc. Imp. Acad. Tokyo 18 (1942), 553–554.

[3] Grätzer. G, General Lattice Theory, Academic Press Inc.1978.

[4] Grätzer .G, G. Birkhauser, Boston, Basel and Berlin ,The Congruence of a finite Lattice, ISBN-13978-0-8176-3224-3

[5] Grätzer. G, Congruence Lattice 101' – Theoretical Computer Science 217 (1999) 279-289.