International Journal of Mathematical Education.
ISSN 0973-6948 Volume 14, Number 1 (2024), pp. 13-20
© Research India Publications
http://www.ripublication.com

On Continuity of ε - Nearest and ε - Farthest Point Mappings

[1] Meenu Sharma* and [2]T.D.Narang

[1]Principal A.S. College for Women, Khanna – 141401 Punjab
[2] Department of Mathematics, Guru Nanak Dev University, Amritsar -143005 Punjab

Abstract

Let G be a non empty subset of a metric space (X, d), $x \in X$ and $\varepsilon > 0$. An element $g_0 \in G$ is called an ε - nearest point to x if $d(x, g_0) \leq d(x ,G)+\varepsilon$, where $d(x ,G) = \inf\{d(x .g) : g \in G\}$. The set of all ε- nearest points to x in G is denoted by $P_{G,\, \varepsilon}(x)$. For a non - empty bounded subset K of a metric space (X ,d) and $\varepsilon>0$, an element $k_0 \epsilon K$ is called an ε- farthest point to $x \in X$ if $d(x , k_0) \geq \delta(x , K) -\varepsilon$, Where $\delta(x ,K)= \text{sub}\{d(x, k): k \in K\}$. The set of all ε - farthest points to x in K is denoted by $F_{K,\, \varepsilon}(x)$ .The set- valued mapping $P_{G,\varepsilon}$(Respectively $F_{K,\varepsilon}$) which takes each element $x \in X$ to the set $P_{G,\varepsilon}(x)$ (respectively $F_{K,\varepsilon}(x)$) is called ε -nearest(respectively ε-farthest) point map. In this paper, we discuss the continuity of these two maps. The underlying spaces are metric spaces.

For a non- empty subset G of a metric space (X,d) , $x \in X$ and $\varepsilon>0$, we say that $g_0 \in G$ is an ε -approximation or good approximation to x if $d(x, g_0) \leq d(x, g) + \varepsilon$ for all $g \in G$ (for $\varepsilon = 0$, such a g_0 is a best approximation to x in G) . R.C. Buck [1] was the first to introduce and discuss this notion in the theory of nearest points (best approximation). Analogously, the notion of ε - farthest points was introduced and discussed by Narang [4] in the theory of farthest points as under:

For a non -empty bounded subset K of a metric space (X ,d) , $x \in X$ and $\varepsilon>0$, we say that $k_0 \in K$ is an $\varepsilon -$ farthest point to x if $d(x, k_0), \geq d(x, k)- \varepsilon$ for all $k \in K$ (for $\varepsilon = 0$, such a k_0 is a farthest point to in x in K). The two notions of $\varepsilon -$ approximation and $\varepsilon -$farthest point have been discussed in [1]- [15] and in references cited therein .

In this paper we discuss the continuity of the two maps which take each element of the space to the set of its ε - nearest and ε - farthest points respectively. The underlying spaces are metric spaces. We start with recalling few definitions and results needed in this sequel.

Let G be a non - empty (K be a non- empty bounded) subset of a metric space (X , d), $x \in X$ and $\varepsilon>0$. An element $g_0 \in G$($k_0 \in K$) is called ε –approximation or ε - nearest point or good approximation (ε - farthest point or nearly farthest point) to x in G (in K) if $d(x, g_0) \leq d(x, G)+ \varepsilon$, where $d(x, G) = \inf\{d(x, g): g \in G\}$ (if $d(x, k_0) \geq \delta(x, K)- \varepsilon$ where $\delta(x, K) =\sup\{d(x, k) : k \in K\}$) . The set of ε- approximation to x in G (ε- farthest points to x in K) is denoted by $P_{G,\varepsilon}(x)$($F_{K\,\varepsilon}(x)$) i.e.

$$P_{G,\varepsilon}(x) = \{g_0 \in G : d(x.g_0) \leq d(x .G)+\varepsilon\}$$

$$F_{K,\varepsilon}(x) = \{k_0 \in K : d(x.k_0) \geq \delta(x .K)-\varepsilon\}$$

Since elements of ε- approximation (ε-farthest point) always exist for $\varepsilon > 0$, $P_{G,\varepsilon}(x)$($F_{K,\varepsilon}(x)$) is always a non-empty subset of G (of K) . The set $G(K)$ is said to be

(i) ε- proximinal (ε -remotal) if $P_{G,\varepsilon}(x)$($F_{K,\varepsilon}(x)$) is non-empty for each $x \in X$.

(ii) ε- uniquely proximinal (ε – uniquely remotal) if $P_{G,\varepsilon}(x)$($F_{K,\varepsilon}(x)$) is a singelton for each $x \in X$.

A sequence $\langle g_n \rangle$ in G ($\langle k_n \rangle$ in K) is said to be ε- minimizing (ε- maximizing) sequence for x if $\lim\limits_{n\to\infty} d(x.g_n) \leq d(x, G) + \varepsilon$ ($\lim\limits_{n\to\infty} d(x, k_n) \geq (x,K)-\varepsilon$).

The set $G(K)$ is said to be ε- approximatively compact (ε- nearly compact) if for each $x \in X$, each ε- minimizing (ε- minimizing) sequence has a subsequence converging to an element of G(of K).

The multi valued mapping $P_{G,\varepsilon}(F_{k,\varepsilon})$ which takes each element $x \in X$ to the set $P_{G,\varepsilon}(x)$($F_{K,\varepsilon}(x)$) is called ε- nearest point map or ε- min projection or ε- metric projection (ε- farthest point map or ε- max-projection or ε- metric antiprojection) .

Remark: For ε=0, we have the corresponding concepts in the theory of nearest (farthest) points. Let (X ,d) and (Y ,d') be two metric spaces and 2^Y denote the collection of all bounded closed subsets of Y.

A multi-valued mapping f: $X\to 2^Y$ is is called (see [14] p.386) upper semi- continous (respectively lower semi continous) if the set

$$\{x \in X : f(x) \subseteq M\}$$

is open for every open M ⊆ Y (respectively closed for every M ⊆ Y) or, what is equivalent , if the set

$$\{x \in X : f(x) \cap N \neq \emptyset\}$$

is closed for every closed $N \subseteq Y$ (respectively open for every $N \subseteq Y$)

f is said to be upper (K) semi - continous (respectively lower (K) semi - continous) (see [14], p. 388) if the relations $\lim_{n\to\infty} x_n = x$, $y_n \in f(x_n)$, $n = 1,2,....$, $\lim_{n\to\infty} y_n = y$ imply $y \in f(x)$ (respectively, if the relations $\lim_{n\to\infty} x_n = x$, $y \in f(x)$ imply the existence of a sequence y_n with $y_n \in f(x_n)$, $n = 1,2,...$ such that $\lim_{n\to\infty} y_n = y$).

It is well known (see [14], p.389) that every upper (lower) semi continous mapping is upper (lower) (K) – semi- continous and, if Y is compact, then the converse is also valid. Moreover, if f is single -valued, then the concepts of upper (lower) semi- continuity and continuity coincide.

The following results on ε -nearest points and ε- farthest points are known in metric spaces.

(i) By the definition of $d(x, G)$ ($\delta(x, K)$) elements of $\varepsilon -$ approximation (ε- farthest point) always exist for every $\varepsilon > 0$ and so every non- empty set G (every non-empty bounded set K) is always ε- proximinal (ε- remotal) .

(ii) [12] $P_{G,\varepsilon}(x)$($F_{K,\varepsilon}(x)$) is a bounded subset of X.

(iii) [12],[15] $P_{G,\varepsilon}(x)$($F_{K,\varepsilon}(x)$) is a closed subset of X if $G(K)$ is a closed subset of X.

(iv) [12] $P_{G,\varepsilon}(x)$ is a compact subset of X if G is a boundedly compact subset of X.

(v) [13] $F_{K,\varepsilon}(x)$ is a compact subset of X if K is a bounded closed ε- nearly compact subset of X

(vi) [12] $P_{G,\varepsilon}(x) = G \cap B[x, d(x, G) + \varepsilon]$, where $B[x, r]$ denotes closed ball with centre x and radius r .

(vii) [13] $F_{K,\varepsilon}(x) = K \cap B(x, \delta(x, K) - \varepsilon)^c$, where $B(x, r)^c$ denotes complement of the open ball with centre x and radius r.

(It was inadverntly written in [13] that $F_{K,\varepsilon}(x) = K \cap B[x, \delta(x, K) - \varepsilon]^c$)

(viii) [12]$g_0 \in P_{G,\varepsilon}(x)$ if and only if $g \in B(x, d(x, g_0) - \varepsilon)^c$ for each $g \in G$ i.e. $G \cap B(x, d(x, g_0) - \varepsilon) = \emptyset$.

(ix) [13] $k_0 \in F_{K,\varepsilon}(x)$ if and only if $k \in B[x, d(x, k_0) + \varepsilon]$ for each $k \in K$ i.e. $K \cap B[x, d(x, k_0) - \varepsilon]^c = \emptyset$.

The following theorem deals with the upper (K) – semi- continuity of the ε- nearest point map.

Theorem 1. If G is an arbitrary closed subset of a metric space (X, d) , then for any $\varepsilon > 0$, the mapping $P_{G,\varepsilon}: X \to 2^G$ is upper (K) - semi - continous , where 2^G is the collection of all bounded closed subsets G.

Proof: By the definition, $P_{G,\varepsilon}(x) \neq \emptyset$ for each $x \in X$ i.e. G is ε- proximinal . Since G is closed, $P_{G,\varepsilon}(x)$ is closed . Moreover, $P_{G,\varepsilon}(x)$is bounded and so $P_{G,\varepsilon} : X \to 2^G$.

Let $\langle x_n \rangle$be a sequence in X such that $\lim\limits_{n\to\infty} x_n = x$, $y_n \in P_{G,\varepsilon}(x_n)$,$n$ =1,2,.., and $\lim\limits_{n\to\infty} y_n = y$. Consider

$$d(x_n, y_n) \leq d(x_n, G) + \varepsilon.$$

This gives

$\lim\limits_{n\to\infty} d(x_n, y_n) \leq \lim\limits_{n\to\infty} d(x_n, G) + \varepsilon$

i.e. $d(x,y) \leq d(x, G) + \varepsilon$.

This implies that $y \in P_{G,\varepsilon}(x)$, and hence $P_{G,\varepsilon}$ is upper(K)- semi- continous. Concerning the upper semi- continuity of the map $P_{G,\varepsilon}(x)$, we have

Theorem 2. If G is an ε- approximatively compact closed subset of a metric space (X,d) , then the ε- projection map $P_{G,\varepsilon}$ maps X into 2^G and is upper semi- continous .

Proof: Let N be an arbitrary closed subset of G. We shall show that the set G

$$A = \{x \in X: P_{G,\varepsilon}(x) \cap N \neq \emptyset\}$$

is a closed set. Let x be a limit point of A. Then there exists a sequence $\langle x_n \rangle$ in A such that $\langle x_n \rangle \to x$. Since $x_n \in A$, there exists $y_n \in P_{G,\varepsilon}(x_n) \cap N$, n = 1,2,... i.e .$d(x_n, y_n) \leq d(x_n, G) + \varepsilon$ and $y_n \in N$, n =1,2,…

Consider

$d(x, y_n) \leq d(x, x_n) + d(x_n, y_n)$

$\leq d(x, x_n) + d(x_n, G) + \varepsilon$

$\leq 2d(x, x_n) + d(x, G) + \varepsilon$ **(1)**

Taking limit as n→∞, we obtain

$\lim\limits_{n\to\infty} d(x, y_n) \leq d(x, G) + \varepsilon.$

Since G is ε- approximatively compact, there exists a subsequence $\langle y_{n_i} \rangle$ of $\langle y_n \rangle$ such that $\langle y_{n_i} \rangle \to y_0 \in G$, and so

$$d(x, y_0) \leq d(x, y_{n_i}) + d(y_{n_i}, y_0)$$
$$\leq 2d(x, x_{n_i}) + d(x, G) + \varepsilon + d(y_{n_i}, y_0) , \quad \text{by (1)}$$

gives $d(x, y_0) \leq \lim\limits_{n\to\infty} 2d(x, x_{n_i}) + d(x, G) + \varepsilon$ i.e. $d(x, y_0) \leq d(x, G) + \varepsilon$. Therefore $y_0 \in P_{G,\varepsilon}(x)$.

Since $y_{n_i} \in N, \langle y_{n_i} \rangle \to y_0$ and N is is closed , $y_0 \in N$ and so, $y_0 \in P_{G,\varepsilon}(x) \cap N$. Consequently, $x \in A$ and hence $P_{G,\varepsilon}$ is upper semi - continous.

Corollary 1: If G is an ε -approximatively compact, ε -uniquely proximinal closed subset of a metric, space (X, d) then the map $P_{G,\varepsilon}: X \to G$ is continous.

Proof: Since G is ε–uniquely proximinal, $P_{G,\varepsilon}$ is single -valued and hence $P_{G,\varepsilon}$ is continous.

In the particular case when ε= 0, we obtain

Corollary 2:

(i) (see [14], p.386) If G is an approximatively compact subset of a metric space (x, d), then the metric– projection is upper– semi– continous .

(ii) (see [14], p.390) If G is an approximatively compact Chebyshev subset of a metric space (X, d), then the metric– projection P_G is continous.

Concerning the upper(K) –semi continuity of the ε –farthest map, we have

Theorem 3: Let K be a bounded closed subset of a metric space (X, d), then for any ε>0 the mapping $F_{K,\varepsilon} : X \to 2^K$ is upper(K) semi continuous .

Proof: By the definition, K is ε- remotal i.e. the set $F_{K,\varepsilon}(x) \neq \phi$ for every $x \in X$. Since K is closed and bounded, $F_{K,\varepsilon}(x)$ is closed and bounded. So $F_{K,\varepsilon}$ maps X into 2^K.

Let $\langle x_n \rangle$ be a sequence in X such that $\lim\limits_{n\to\infty} x_n = x$, $y_n \in F_{K,\varepsilon}(x_n)$, $n = 1,2,\ldots$ and $\lim\limits_{n\to\infty} y_n = y$ i.e.

$$d(x_n, y_n) \geq \delta(x_n, K) - \varepsilon \text{ and } \langle y_n \rangle \to y$$

Therefore

$$\lim_{n\to\infty} d(x_n, y_n) \geq \lim_{n\to\infty} \delta(x_n, K) - \varepsilon \text{ and } \langle y_n \rangle \to y \text{ , and so}$$

$$d(x, y) \geq \delta(x, K) - \varepsilon .$$

This gives $y \in F_{K,\varepsilon}(x)$ and hence $F_{K,\varepsilon}$ is upper(K) -semi -continous .

The following theorem deals with the upper semi- continuity of ε–farthest point map.

Theorem 4. Let K be a bounded ε- nearly compact closed subset of a metric space (X, d), then the ε – antimetric projection map $F_{K,\varepsilon}: X \to 2^K$ is upper semi- continous .

Proof: By the definition, K is ε- remotal i.e. $F_{K,\varepsilon}(x) \neq \phi$. Moreover, $F_{K,\varepsilon}(x)$ is closed as K is closed, and bounded as K is bounded and so $F_{K,\varepsilon}$ maps X into 2^K .

Let W be an open subset of X and $M = \{z \in X, F_{K,\varepsilon}(z) \subseteq W\}$. We claim that M is open. Let $\langle z_n \rangle$ be a sequence in X such that $z_n \to z_0$, where $z_0 \in$ M. We show that $\langle z_n \rangle$ is eventually in M. Suppose it is not true, then there exists a subsequence $\langle z_{n_i} \rangle$ of $\langle z_n \rangle$ such that $F_{K,\varepsilon}(z_{n_i}) \cap W^c \neq \phi$. Let $y_{n_i} \in F_{K,\varepsilon}(z_{n_i}) \cap W^c$, then

$$d(z_{n_i}, y_{n_i}) \geq \delta(z_{n_i}, K) - \varepsilon$$

Since $|d(z_{n_i}, y_{n_i}) - d(z_0, y_{n_i})| \leq d(z_{n_i}, z_0)$, we obtain

$$\begin{aligned} d(z_0, y_{n_i}) &\geq d(z_{n_i}, y_{n_i}) - d(z_{n_i}, z_0) \\ &\geq \delta(z_{n_i}, K) - \varepsilon - d(z_{n_i}, z_0) \end{aligned}$$

and so $$\lim_{n\to\infty} d(z_0, y_{n_i}) \geq \delta(z_0, K) - \varepsilon \qquad \textbf{(1)}$$

i.e. $\langle y_{n_i} \rangle$ is a maximizing sequence in K for z_0.

Since K is ε- nearly compact, $\langle y_{n_i} \rangle$ has a subsequence $\langle y_{n_i} \rangle \to y_0 \in$ K. Then (1) implies

$$d(z_0, y_0) \geq \delta(z_0, K) - \varepsilon, \quad y_0 \in F_{K,\varepsilon}(z_0)$$

Since W^c is closed, $y_0 \in W^c$. So $y_0 \in F_{K,\varepsilon}(z_0) \cap W^c$. Therefore, $z_0 \notin$ M, a contradiction. Hence is $\langle z_n \rangle$ is eventually in M and hence M is open and consequently, $F_{K,\varepsilon}$ is upper semi- continous.

Corollary 1: If K is a bouded ε–nearly compact, ε– uniquely remotal closed subset of a metric space, (X, d), then ε– farthest point map $F_{K,\varepsilon}: X \to K$ is coninous.

Proof: Since K is ε– uniquely remotal, $F_{K,\varepsilon}$ is single valued and hence coninous .

For the particular case ε= 0, we obtain

Corollary 2: [3]

(i) If K is a nearly compact set in a metric space (X, d), then the farthest point mapping F_K maps X into 2^K and is upper semi- continous.

(ii) If K is a nearly compact uniquely remotal subset of a metric space (X, d), then the farthest point map is continous.

Future Directions

It will be interesting to discuss lower semi- continuity of ε– nearest and ε– farthest point maps. As a counter part to the concept of ε– approximation, the concept of ε– coapproximation can be defined as under (see [12] and references cited therein) :

For a subset G of a metric space (X,d) and $\varepsilon > 0$, an element $g_0 \in G$ is called ε-coapproximation to $x \in X$ if $d(g_0, g) \le d(x, g) + \varepsilon$ for all $g \in G$. The set of all ε-coapproximations to x in G is denoted by $R_{G\,\varepsilon}(x)$.

The continuity of the map which takes each element $x \in X$ to its set of ε-coapproximation in G, remains to be investigated.

It may be remarked that the concepts of ε- cofarthest points ($d(k_0, k) \ge d(x, k) - \varepsilon$ for all $k \in K$) and so of co-farthest points (taking ε=0) are not meaningful.

References

[1] R.C. Buck , Applications of duality in approximation theory , Approximation of Functions , Elsevier, Amsterdam (1965), 27 - 42 .

[2] M. Janc, Nearly farthest points in normed linear spaces, Mat. Vesnik, 39 (1987), 309 – 319.

[3] T.D.Narang,Nearly Compact sets and farthest point map,Indian J. Pure Appl. Math, 9(1978),116 - 118

[4] T.D.Narang, Metric antiprojections and Characterizations of farthest points , J.Korea Soc. Math, ε– Education , 21(1983),19 – 21

[5] T.D.Narang, On farthest and nearly farthest points, Mat. Vesnik, 44(1992),105 - 111 .

[6] T.D.Narang , On good approximation , J. Scientific Research , 49(1999), 25 - 43 .

[7] T.D.Narang, On ε– fixed points and ε– approximation, Math Today, 22(2006),15 - 19

[8] T.D.Narang, On simultaneous characterization of sets of elements of good approximation in metric spaces, African Diaspora J. Math, 10(2014),23-30 .

[9] Meenu Sharma and T.D.Narang, On ε– Birkhoff orthogonality and ε– near best approximation, J. Korea Soc. Math. Education 8 (2001), 153 - 162.

[10] Meenu Sharma and T.D.Narang, On invariance of ε– orthogonality , ε– approximation and ε– coapproximation in metric linear spaces, Int. J. Engg. Sci. 4 (2015), 20 - 25.

[11] Meenu Sharma and T.D.Narang, On ε– approximation, International J. Engineering and Science,11 (2022) , 12 - 19.

[12] Meenu Sharma and T.D.Narang, On ε– coapproximation, Indian J. Mathematics Research, 12 (2023) , 1- 6.

[13] Meenu Sharma and T.D.Narang, On ε– farthest points and their invariance, International J. Mathematical Education, 13 (2023), 9-14 .

[14] Ivan Singer, Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces, Springer -Verlag, New York, 1970.

[15] S.M.Vaezpour , R.Hasani and H.Mazaheri , On the ε– best coapproximation , International ε– Mathematical Forum , 2 (2007) , 2599 - 2606 .