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A Comprehensive Analysis of Lateral Scaffolding Techniques in Enhancing Student Math Problem Solving

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Abstract

Researchers are still interested in teaching problem solving techniques that are both general and domain specific. The complexity of the interplay among various factors makes it challenging to comprehend the problem-solving process in mathematics. One approach to grasp this is to give the student a specific challenging problem to work through, which will enable the researcher to figure out at least some of the influencing factors. This paper presents our findings on how individuals respond to a math problem, with the inclusion of lateral scaffolding's during their problem-solving journey. Examining which scaffolds learners choose to use, and when and how they implement them, reveals the factors that can either encourage or hinder their efforts. Results analysis suggests that a targeted instructional method may be necessary to strengthen skill sets, and that embedding this approach within a well-designed math problem could cultivate effective problem-solving skills.

Keywords Lateral Scaffolding · Problem Solving · Mathematical Thinking · Maths Problems

1. Introduction

Education is the journey through which individuals are empowered to realise their full potential while also preparing them to become productive members of society. Through the processes of teaching and learning, individuals acquire essential knowledge and skills that significantly enhance their problem-solving abilities. This kind of development not only helps one grow personally, but it also gives him or her, the tools they need to overcome obstacles and give back to their communities. Mathematics is one of the subjects of study that plays a significant part in the educational system. As a result, beginning in elementary school, all children are taught mathematics as a formal and an important subject. Students often find math straightforward with formula-based questions but can struggle otherwise. One of the skills students might acquire from solving math problems includes reasoning, as well as the methods for tackling more complex challenges and the logic to improve their solutions to familiar problems.

Lateral Scaffolding Method: Lateral Scaffolding is an educational approach created by Pieter van der Merwe. This method maintains the original content of the learning materials, focusing instead on altering the manner in which the study process is organized and executed. In straightforward terms, the core concepts, sub-themes, and fine points of the content are recognized and subsequently organized into a more holistic learning framework. This method enables the learner to grasp the overarching ideas and, based on the material presented, fosters a deeper comprehension of how the fine/detailed aspects are rooted in the sub-themes, which in turn are interconnected with the core concepts. Its significant educational benefits are:

1) The “bigger picture” approach
2) Recapitulation of Learning Material
3) The Art of Making Associations
4) Better Retention of Content
5) Immediate Feedback of Performance

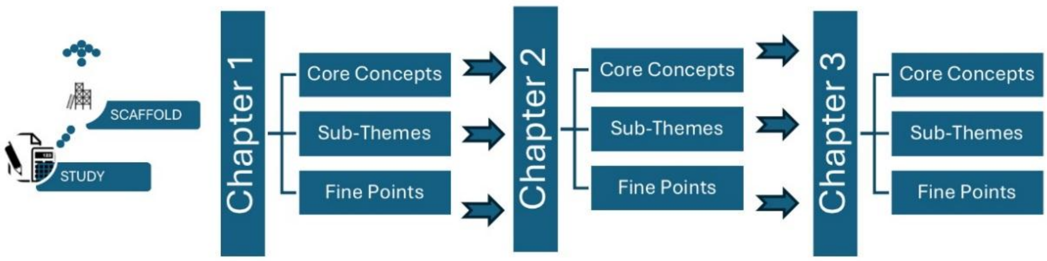

Figure. 1 Diagrammatic Representation of Lateral Scaffolding Method

Literature Survey

The existing literature on these approaches underscores their effectiveness in promoting deeper mathematical understanding, critical thinking, and adaptive expertise. Bunce and Heikkinen (1986) explored the impact of explicit problem-solving strategies, demonstrating that structured scaffolding significantly improved mathematical chemistry achievement. Similarly, Frank, Baker, and Herron (1987) argued for the selective use of algorithms, emphasizing that rote application may hinder conceptual understanding in mathematics. The focus on understanding students’ mathematical thought processes has been central in the work of

Carpenter et al. (1989), who highlighted the role of teachers in tailoring instructional strategies based on students' problem-solving approaches. Cobb et al. (1991) extended this notion by presenting classrooms as collaborative learning environments where teachers and students engage in co-constructing mathematical knowledge, emphasizing the social dimension of learning. Cobb et al. (1997) further delved into reflective discourse as a scaffolding mechanism, illustrating how collective reflection facilitates shared understanding. This theme of communication and representation in mathematics was expanded by Cobb, Yackel, and McClain (2000), who discussed the interplay of discourse, tools, and instructional design in fostering problem-solving skills. The qualitative aspects of problem-solving were examined by Chi (1997), who provided a framework for analyzing students' verbal data during problem-solving tasks. This approach complements the work of van Oers (2000), who posited that engaging students in meaningful mathematical activities encourages active participation and deeper comprehension.

Moyer and Milewicz (2002) introduced the significance of diagnostic questioning in scaffolding, categorizing effective questioning strategies to assess and guide students' reasoning processes. Pretz, Naples, and Sternberg (2003) identified key stages of problem-solving, including recognizing and defining problems, which align with lateral scaffolding techniques aimed at building adaptive reasoning. Kazemi and Franke (2004) emphasized teacher collaboration and the use of student work to drive collective inquiry, a process that Steinberg, Empson, and Carpenter (2004) found to be pivotal in reshaping teacher practices and fostering student-centered learning environments. Similarly, Wallach and Even (2005) investigated the complexity of interpreting students' verbal and non-verbal cues, highlighting the nuanced role of teacher understanding in scaffolding. Carson and Bloom (2005) introduced a multidimensional framework for problem-solving, advocating for iterative scaffolding processes that adapt to students' evolving needs. Singh (2007) reinforced this perspective by categorizing problems to facilitate targeted instructional strategies, while Yeşildere and Türnüklü (2007) examined mathematical reasoning as a core component of scaffolding techniques. Jonassen (2010) outlined critical research issues in problem-solving, calling for integrative scaffolding approaches that address cognitive and emotional aspects. Schoenfeld (2013) reflected on the theoretical underpinnings of problem-solving, underscoring the need for dynamic scaffolding frameworks that evolve with students' skills. Palinussa (2013) demonstrated how culturally contextualized scaffolding fosters critical mathematical thinking, while Hu and Rebello (2014) and Zuza et al. (2015) explored domain-specific scaffolding techniques to address students' epistemological and conceptual challenges.

2. Methodology

This type of research is descriptive qualitative and focuses on explaining phenomena through an in-depth collection of factual information. In our study, we have adopted a two phase approach. We have obtained and analyzed responses to well framed, unambiguous math problems designed in Multiple Choice Question (MCQ) format and then carried out exploratory interviews to further inquire comprehensively the students' problem solving methodologies. A set of math problems in MCQ format was administered to 250 undergraduate students, with each question's options designed to fulfill clear objectives. The students' answers were analyzed. This paper outlines the findings of our inquiry into a selected sample problem in Linear Algebra. The answers to the MCQ test presents insights that were utilized in crafting the interview protocol. Designed with the right scaffolding, the interview protocol aims to uncover the fine structure of their knowledge frameworks. During the next stage, the validation interview serves as the initial step, aimed at confirming if a question conveys the meaning the interviewer wishes to express to the student. Drawing on this input, the questions were modified where necessary, and the scaffolding's were refined. This process established a groundwork for the data interviews. We have interviewed 15 students with this problem. Every student underwent a one-on-one interview. Scaffolding's were given to facilitate their advancement toward solving the problem.

The study included students aged nineteen to twenty-one. All of them had studied Linear Algebra in their undergraduate curriculum. The questions they faced were about topics they had formally learned in classroom.

Structural Design for the question: Given $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. What are the eigenvalues of A ?

a) $\lambda = 4, 0, 0, 0$
b) $\lambda = 4, 0, 2, -2$
c) $\lambda = 4, 0, 1, -1$
d) $\lambda = 4, 4, 0, -4$

Eigenvalues are scalar values associated with a square matrix that measure how a matrix transforms a vector. Let us consider a square matrix A of order n and v be an eigenvector, then λ is a scalar quantity represented in the following way: $Av = \lambda v$. Here, λ is considered to be the eigenvalue of matrix A. The above equation can also be written as: $(A - \lambda I)v = 0$, where I is the identity matrix of order n, and 0 is a null matrix of order n. The eigenvalues are obtained by solving the characteristic equation

$$|A - \lambda I| = 0$$

which implies that

$$\begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

on solving the above determinant, we obtain $\lambda = 4, 0, 0, 0$ as eigenvalues of A.

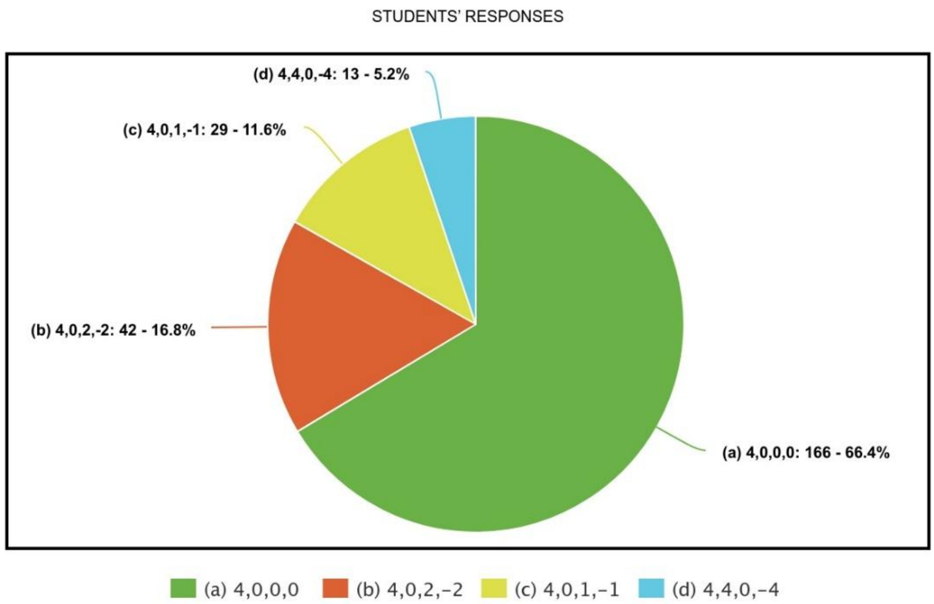

Figure. 2 Students Responses Pie Chart

Level 1: Pilot Study. We observe that 66.4 % of the students chose the correct option. The nature of the distribution of the responses indicates a reasonable easiness in the solving the question correctly and the rationale for the question is lucid. This indicates the lack of "forming associations" between the topics. The review of the MCQ responses identifies several tasks. A significant one is the necessity to assess the challenges in translating intricate details into well-defined concepts.

Level 2: Personal Interview. There is a strong inclination among students to find the relevant equation, so any indication in this direction provides substantial scaffolding. The weakest scaffolding was presented first in the interview protocol's design. The classification of scaffolding as weak or strong is not absolute; the order was developed from the analysis of MCQ responses and validation interviews.

The arrangement of the scaffoldings was determined based on the limpidity of the concepts, meaning the most straightforward information was prioritized initially.

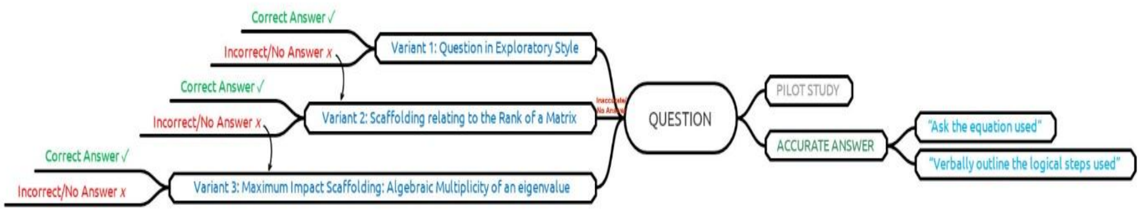

Figure. 3 Mind map representing scaffolding strategy management

Variant 1: Question posed in exploratory style. To start, we framed the question in the exploratory style as provided below.

Consider a square matrix A of order n and a nonzero vector v of length n. If multiplying A with v simply scales v by a factor of λ, where λ is a scalar, then v is called an eigenvector of A, and λ is the corresponding eigenvalue. This relationship can be expressed as: $Av = \lambda v$.

Given $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. What are the eigenvalues of A ?

Among the fifteen students, only two managed to solve the problem effectively. They grabbed the relationship expressed $Av = \lambda v$, then found out the correct option using the characteristic equation. The remaining students were still in a confused state.

This type of observation from the student body is fairly common. Clearly, there is a need to address the gap between the correct conceptual framework and the mental models that students possess.

Variant 2: Question posed with Scaffolding explaining (Hint relating the rank of a matrix)

We restated the question in this form for students who did not succeed in solving the problem.

Given $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and the rank of any matrix is the number of non-zero eigenvalues. What are the eigenvalues of A ?

a) $\lambda = 4, 0, 0, 0$
b) $\lambda = 4, 0, 2, -2$
c) $\lambda = 4, 0, 1, -1$
d) $\lambda = 4, 4, 0, -4$

Of the thirteen students remaining, there were seven who successfully solved the problem.

The rank of A is equal to the number of non-zero rows in the row echelon form of A. The rank of matrix A can be easily determined to be 1 by the method of Row Reduction. The hint given is straightforward, the number of non-zero eigenvalues is one, that is 4. Undoubtedly, out of the options given, option (a) has the accurate answer. The skill to form associations for a different context is not an automatic learning result; it demands careful restructuring during the teaching process.

Variant 3: Question with Maximum Impact Scaffolding (Hint relating Algebraic Multiplicity $A.M.$ of an eigenvalue)

Given $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and $A.M.(0) = 3$. What are the eigenvalues of A ?

a) $\lambda = 4, 0, 0, 0$
b) $\lambda = 4, 0, 2, -2$
c) $\lambda = 4, 0, 1, -1$
d) $\lambda = 4, 4, 0, -4$

Four out of the remaining six students were able to successfully solve the problem. The algebraic multiplicity of an eigenvalue is the number of times it appears as a root of the characteristic polynomial. $A.M.(0) = 3$ directly indicates that the algebraic multiplicity of 0 is three, exposing that the right option is (a) since 0 is repeated thrice only in this very option.

Students who knew the definition of algebraic multiplicity of an eigenvalue could answer it successfully. Only two students were unsuccessful to solve the problem. Inspection of strategies utilized by those students who solved the problem accurately in their interviews:

We sought to look into both the previously mentioned parameters and the techniques used by successful solvers. Additionally, we intended to explore the knowledge structure relevant to solving the problem, beginning with the question below:

"What equation is used to find the eigenvalues of a square matrix?"

The answers were … "Characteristic equation."

We inquire about this because we theorize that a newcomer would address a problem by randomly searching for equations. All the students who received this question could answer it.

In the next phase, we questioned the students about their ability to verbally outline the logical steps that they used to find the answer. "In this problem, we first find the characteristic equation of A, then on solving it, we get the eigenvalues as 4 and 0. The eigenvalue 0 is repeated three times."

3. Results and Discussions

The investigations demonstrate a strategy for assessing the parameters that impede the resolution of math problems. Current instructional methods generally focus on fostering a conceptual understanding and computational rigor. Nevertheless, students frequently fail to elaborate on their ideas beyond definitions and statements of laws, particularly in problem-solving contexts. This investigation thoroughly explores the intricate details of student thought processes while solving problems. Structured interviews, paired with carefully designed scaffolding, illuminate specific challenges that often go unnoticed. Consequently, we arrive at the following conclusions.

As an initial approach to problem-solving, it is important to find the most applicable concept for the scenario. However, the weak ties in students' conceptual understanding of the principles greatly hinder their problem-solving abilities. Another vital component that can affect the ability to solve problems is constricted existence of the 'mathematical manipulation' expertise.

This suggests a viable method for reinforcing problem-solving skills focusing on the development of skill sets - "the art of making associations & bigger picture approach" rather than solving a multitude of problems. The appropriate design and direction of efforts to cultivate the necessary skills can then serve as the primary focus for this need.

An essential consideration in such studies is the undeniable influence of the problem solver's domain vulnerability within the context of the problem. A student who lacks clarity in mathematical concepts may not benefit significantly from scaffolding's. Such scaffolding may not stimulate the student to find a solution in these circumstances. A further limitation is that scaffolding's lack uniqueness and cannot be arranged sequentially. The unavailability of such a hierarchical system might weaken a set of scaffolding's which is produced for a question, intended for a learner. Given this possible limitation of the probing methodology, it is hardly feasible to design unique scaffolding's that rely on the system rather than the learner. However, our studies validate the utility of the technique described in this paper.

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