Wavelet Based Approximation Method for Solving Wave and Fractional Wave Equations Arising in Ship Dynamics

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Abstract

In this paper, we have applied the wavelet based approximation method for solving wave-type and fractional wave-type equations arising in ship dynamics. The basic idea of Legendre wavelets method (LWM) is to convert the partial differential equations (PDEs) into a system of algebraic equations, which involves a finite number of variables. Convergence analysis of the Legendre wavelets is also discussed. Finally, we have given the illustrative examples to demonstrate the applicability and validity of the proposed method.

Keywords: Wave-type equation, Legendre wavelets, Laplace transform method.

1. INTRODUCTION

Wavelet analysis, as a relatively new and emerging area in applied mathematical research, has received considerable attention in dealing with partial differential equations and fractional partial differential equations (PDEs). Wavelets analysis establish a connection with fast approximation algorithms [5-10, 14-17, 38, 40-42, 43, 45]. In recent years, the wave and fractional wave equations play a significant role in Mathematical Physics and many scientific applications. One-dimensional wave-type equation is not only the simplest second order hyperbolic PDE, but also a development-type equation to describe the phenomenon of wave vibration and wave propagation [1, 30]. Therefore, investigating a rapid and accurate method of solving the wave-type equation is great significant.

This problem includes a vibrating string, vibrating membrane, longitudinal vibrations of an elastic rod or beam, shallow water waves, acoustic problems, shock waves,

sediment transport in rivers and solitons in collisionless plasma [23]. Fractional partial differential equations are generalizations of classical partial differential equations of integer order. In contrast to simple classical systems, where the theory of integer order differential equations is sufficient to describe their dynamics, fractional derivatives provide an excellent and an efficient instrument for the description of memory and hereditary properties of various complex materials and systems. But these nonlinear PDEs are difficult to get their exact solutions. So the numerical methods must be used. The approximation solutions of the wave and fractional wave equations have received considerable attention in the literature and fall into two groups: the analytical methods and the numerical ones. Analytical methods enable researchers to study the effect of different variables or parameters on the function under study easily. Recently, many new iterative methods have been used to solve the wave-type and fractional PDEs, for example, the Adomain decomposition Method [23], the variational iteration method [29, 32], the reduced differential transform method [12, 34], the homotopy perturbation method [2, 3], the homotopy analysis method [31, 33, 35], and other methods [10, 11, 13, 19, 20, 22, 24]. Although there are already exists a huge literature for solving wave and fractional equations.

In wavelet based approximation methods, there are two important ways of improving the approximation of the solutions: increasing the order of the wavelet family and the increasing the resolution level of the wavelet. The main aim of this research work is to show how wavelets and multi-resolution analysis can be applied for improving the method in terms of easy implementability and achieving the rapidity of its convergence. Recently, Hariharan and Kannan [44] reviewed wavelet methods for the solution of reaction-diffusion problems in science and engineering.

There is a growing interest in using wavelets [4-9, 18, 21, 26-28] to study problems, of greater computational complexity. Among the wavelet transform families the Haar and Legendre wavelets deserve much attention. Lepik [25] applied the Haar wavelet method for evolution equations. Recently, Hesameddini and Shekarpaz [36, 37] used the Legendre wavelet method for solving Klein-Gordon equations and second Painleve equation. Razzaghi and Yousefi [4, 6] applied the Legendre wavelet method for variational problems and constrained optimal control problems. Hariharan et al. [14-17] applied the Haar wavelet method for convection-diffusion equation, Reaction-diffusion equation, Non linear parabolic equations, fractional Klein-Gordon equations, sine-Gordon equations and Fisher's equation. Mohammadi and Hosseini [8] had showed a new Legendre wavelet operational matrix of derivative in solving singular ordinary differential equations. Parsian [5] introduced the two dimensional Legendre wavelets and operational matrices of integration. For complete sake of Legendre wavelet is presented in literature [4-9, 18, 36]. Recently, Hariharan and R. Rajaraman [39] established a new coupled wavelet-based method applied to the nonlinear reaction-diffusion equation arising in mathematical chemistry.

This paper is organized as follows: In section 2, the Multi-resolution analysis (MRA), wavelets, Legendre wavelets are briefly described. In section 3, method of solution by the Legendre wavelet method (LLWM) is presented. In section 4, the convergence

analysis is discussed. Several numerical examples are given to demonstrate the effectives of the proposed method in section 5. Concluding remarks are given in section 6.

2. MULTI-RESOLUTION ANALYSIS (MRA)

2. 1 Scaling Functions and Multiresolution analysis

A function $\phi(x) \in L^2(R)$, is called a scaling function that generates a multiresolution analysis (MRA) in the subspaces $V_{n-1}, V_n, V_{n+1}, ...,$ if the following conditions are satisfied.

i) $V_j \subset V_{j+1}, \ \forall j;$

ii) $f(x) \in V_n \Leftrightarrow f(2x) \in V_{n+1};$

iii) $f(x) \in V_n \Leftrightarrow f(x+2^{-n}) \in V_n;$

iv) $\lim\limits_{n \to \infty} V_n = \bigcup\limits_n V_n$ is dense in $L^2(R);$

v) $\lim\limits_{n \to -\infty} \bigcap\limits_n V_n = \{0\};$

vi)   The set $\{\phi(x-k)\}_{k \in Z}$ forms a Riesz or unconditional basis for V_0, i. e, there exist constants A and B, with $0 < A \leq B < \infty$, such that,

$$A \sum_{k \in Z} |c_k|^2 \leq \left\| \sum_{k \in Z} c_k \phi(x-k) \right\|_2^2 \leq B \sum_{k \in Z} |c_k|^2$$

for any sequence $\{c_k\} \in l_2$, the subspace of all square summable sequences ($A = B = 1$ for an orthonormal basis).

A scaling function, $\phi(x)$, and a set of related coefficients, $\{p(k)\}_{k \in Z}$, are constructed such that they satisfy the so-called two-scale relation or refinement equation,

$$\phi(x) = \sum_k p(k) \phi(2x-k)$$

and some additional conditions. We say that the scaling function $\phi(x)$ has compact support if and only if finitely many coefficients $p(k)$ are non-zero.

Translations of the scaling function, $\{\phi(x-k)\}$, form a Riesz or unconditional basis of a subspace $V_0 \subset L^2(R)$. Furthermore, through translation of ϕ by a factor of 2^n

and dilation by a factor of $k.2^{-n}$, a Riesz basis, $\{\phi_{n,k}(x)\}_{k \in Z}$, is obtained for the subspace

$V_n \subset L^2(R)$, where

$$\phi_{n,k}(x) = 2^{\frac{n}{2}} \phi(2^n x - k)$$

corresponding to resolution level n. Thus, the scaling function, $\phi(x)$, generates a set of bases for a sequence of nested subspaces of $L^2(R)$, and tends to $L^2(R)$ as the resolution level n, goes to infinity.

2. 2 Wavelets.

Wavelets are the family of functions which are derived from the family of scaling function $\{\emptyset_{j,k:} \ k \in Z\}$ where:

$$\emptyset(x) = \sum_k a_k \ \emptyset(2x - k) \tag{1}$$

For the continuous wavelets, the following equation can be represented:

$$\Psi_{a,b}(x) = |a|^{\frac{-1}{2}} \Psi\left(\frac{x-b}{a}\right) \quad a, b \in R, a \neq 0. \tag{2}$$

Where a and b are dilation and translation parameters, respectively, such that $\Psi(x)$ is a single wavelet function.

The discrete values are put for a and b in the initial form of the continuous wavelets, i.e.:

$$a = a_0^{-j}, \ a_0 > 1, \ b_0 > 1, \tag{3}$$

$$b = k b_0 a_0^{-j}, \quad j, k \ \in Z. \tag{4}$$

Then, a family of discrete wavelets can be constructed as follows:

$$\Psi_{j,k} = |a_0|^{\frac{1}{2}} \Psi(2^j x - k), \tag{5}$$

So, $\Psi_{j,k}(x)$ constitutes an orthonormal basis in L^2 (R), where $\Psi(x)$ is a single function.

2. 3 Legendre Wavelets and its properties [38]

The Legendre wavelets are defined by

$$\psi_{nm}(t) = \begin{cases} \sqrt{m + \frac{1}{2}} \, 2^{\frac{k}{2}} L_m(2^k t - \hat{n}), & \text{for} \quad \frac{\hat{n}-1}{2^k} \leq t \leq \frac{\hat{n}+1}{2^k} \\ \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Where m = 0, 1, 2, …, M-1 and k= 1, 2, …, 2^{j-1}. The coefficient $\sqrt{m + \frac{1}{2}}$ is for orthonormality, then by (4), the wavelets $\Psi_{k,m}(x)$ form an orthonormal basis for $L^2[0, 1]$ [5, 11, 12]. The Legendre polynomials are in the following way:

$$p_0 = 1,$$
$$p_1 = x,$$
$$p_{m+1}(x) = \frac{2m+1}{m+1} x\, p_m(x) - \frac{m}{m+1} p_{m-1}(x).$$

Here {$p_{m+1}(x)$ } are the orthogonal functions of order m, which is named the well-known shifted Legendre polynomials on the interval [0, 1]. Note that, in the general form of Legendre wavelets, the dilation parameter is a = 2^{-j} and the translation parameter is b= n 2^j [4-9].

Theorem 1 [38]:

Let $\Psi(x,y)$ be the two-dimensional Legendre wavelets vector defined in Eq. (2. 9), we have $\frac{\partial \Psi(x,y)}{\partial x} = D_y\, \Psi(x,y),$ (7)

$$Dy = \begin{bmatrix} D & O' & \cdots & O' \\ O' & D & \cdots & O' \\ \vdots & \vdots & \ddots & \vdots \\ O' & O' & \cdots & D \end{bmatrix},$$

where D_y is $2^{k-1,}\, 2^{k'-1}MM`$ x $2^{k-1}2^{k'-1}MM`$ and O', D is MM' x MM'matrix is given as

$$D = \begin{bmatrix} F & O & \cdots & O \\ O & F & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & F \end{bmatrix},$$

in which O and F is M' x M' matrix, and F is defined as follows:

$$F_{r,s} = \begin{cases} 2^{k'}\sqrt{(2r-1)(2s-1)}, & r = 2, \ldots, M'; S = 1, \ldots, r-1; and\ r+s\ is\ odd \\ 0, & otherwise \end{cases} \quad (8)$$

2. 4 Block Pulse Functions

The block pulse functions form a complete set of orthogonal functions which defined on the interval [0, b) is defined by

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m} b \le t < \frac{i}{m}b, \\ 0, & elsewhere \end{cases}$$
       (9)

for i = 1, 2, …, m. It is also known that for any absolutely integrable function f(t) on [0, b) can be expanded in block pulse functions:

$$f(t) \cong \xi^T B_m(t)$$
       (10)

$$\xi^T = [f_1, f_2, \dots, f_m], B_m(t) = [b_1(t), b_2(t), \dots, b_m(t)]$$

where f_i are the coefficients of the block-pulse function, given by

$$f_i = \frac{m}{b} \int_0^b f(t)b_i(t)dt \tag{11}$$

3. METHOD OF SOLUTION

Example 3. 1 We consider the well-known one-dimensional wave equation

$$u_{tt} = \alpha u_{xx}, \ 0 < x < \pi, t > 0 \tag{12}$$

with the boundary conditions

$$\left. \begin{array}{l} u(0,t) = k_1(t), \ u(x,0) = l_1(x) \\ u(1,t) = k_2(t), \ u(x,0) = l_2(x) \end{array} \right\} \tag{13}$$

Taking Laplace transform of both sides of Eq. (12),

$$s^2 L(u) - s. \ u(x, 0) - u_t(x, 0) = L[\ \alpha \ u_{xx}] \tag{14}$$

$$s^2 L(u) = L[\ \alpha \ u_{xx}] + s. \ u(x, 0) + u_t(x, 0) \tag{15}$$

$$L(u) = s^{-2} L[\ \alpha \ u_{xx}] + s^{-1}. \ u(x, 0) + s^{-2} \ u_t(x, 0) \tag{16}$$

Taking inverse Laplace transform, we get

$$u = u(x, 0) + t \ u_t(x, 0) + L^{-1}[\ s^{-2} \ L(\alpha \ u_{xx})] \tag{17}$$

Here

$$L^{-1}\left(s^{-2}L(t^n)\right) = L^{-1}\left(n!s^{-(n+3)}\right) = \frac{t^{n+2}}{(n+1)(n+2)}, \ n = 0,1,2,\dots \tag{18}$$

Then, we have

$$L^{-1}[s^2 \ L(.)] = \int_0^t \int_0^t (.)dt \ dt \tag{19}$$

For our convenience, we define a new operator

$$\prod = L^{-1}[s^2 \ L(.)] \tag{20}$$

Using the Laplace Legendre wavelet method (LLWM), we set

$$u = \ C^T \ \Psi(x,t), u + t \ u_t(x,0) = \ V^T \Psi(x,t) \tag{21}$$

$$C^T = \ V^T + \ (\alpha \ C^T D_x^2) \ P_t^2 \tag{22}$$

Then we introduce the iteration formula as

$$u_{n+1} = u(x,0) + \ t \ u_t(x,0) + \prod(\ \alpha \ \frac{\partial^2 u_n}{\partial x^2}) \tag{23}$$

Now we will start iteration, an initial guess u₀ is required.

Let $u_0 = u(x,0) + tu_t(x,0)$ (24)

The LLWM scheme is given by

$C_{n+1}^T = C_0^T + [\alpha C_n^T D_x^2] P_t^2$ (25)

Example 3. 2 Consider the following equation

$u_{tt} = u_{xx} + f(u), 0 < x < l$ (26)

Taking Laplace transform on both sides of Eq. (26)

$s^2 L(u) - s u(x,0) - u_t(x,0) = L(u_{xx} + f(u))$ (27)

$s^2 L(u) = s^{-1} u(x,0) + s^{-2}u_t(x,0) - s^{-2} L(u_{xx} + f(u))$ (28)

$L(u) = s^{-2} L[\alpha u_{xx}] + s^{-1}.u(x,0) + s^{-2} u_t(x,0) + s^{-2}L(u_{xx} + f(u))$ (29)

By applying the inverse Laplace transform, we obtain

$u = u(x,0) + t u_t(x,0) + L^{-1}\left(s^{-2} L\left(u_{xx} - f(u)\right)\right)$ (30)

Using the traditional Legendre wavelets method,

$u = C^T \Psi(x,t); u(x,0) + t u_t(x,0) = S^T \Psi(x,t)$ (31)

$f(u) = F^T \Psi(x,t)$ (32)

$C^T = S^T + [C^T D_x^2 + F^T] P_t^2$ (33)

We introduce an iteration formula as follows.

$u_{n+1} = u(x,0) + t u_t(x,0) + \Pi(\frac{\partial^2 u_n}{\partial x^2} + f(u_n))$ (34)

To start the iteration, an initial guess of the solution u₀ is required. We let

$u_0 = u(x,0) + t u_t(x,0)$

$C_{n+1}^T = C_0^T + [C_n^T D_x^2 + F^T] P_t^2$ (35)

Example 3. 3 We consider the Klein-Gordon equation arising in quantum physics

$y_{tt} - y_{xx} + b_1 y + b_2 g(y) = f(x,t)$ (36)

$y_{tt} = y_{xx} - b_1 y - b_2 g(y) - f(x,t)$ (37)

Taking Laplace transform of both sides of Eq. (36)

$s^2 L(y) - s y(x,0) - y_t(x,0) = L[y_{xx} - b_1 y - b_2 g(y) + f(x,t)]$ (54)

$s^2 L(y) = s y(x,0) + y_t(x,0) + L[y_{xx} - b_1 y - b_2 g(y) + f(x,t)]$ (38)

$L(y) = s^{-1}y(x,0) + s^{-2}y_t(x,0) + s^{-2}L[y_{xx} - b_1 y - b_2 g(y) + f(x,t)]$ (39)

Using inverse Laplace transform method (ILTM),

$$y = y(x,0) + ty_t(x,0) + L^{-1}\{s^{-2}L[y_{xx} - b_1y - b_2g(y) + f(x,t)]\} \qquad (40)$$

Using the Legendre wavelets,

$$y = C^T \Psi(x,t), y + ty_t(x,0) = S^T\Psi(x,t), g(y) = G^T \Psi(x,t), f(x,t) = F^T\Psi(x,t) \qquad (41)$$

Then

$$C^T = S^T + (C^TD_x^2 - b_1C^T - b_2G^T + F^T)P_t^2 \qquad (42)$$

Here G has a nonlinear relation with C, We solve a non linear algebraic system which make the solution tedious and large computation time. Therefore we use the following iterative formula

$$y_{n+1} = y(x,0) + ty_t(x,0) + \prod(\frac{\partial^2 y_n}{\partial x^2} - b_1y_n + b_2g(y_n) + f(u_n)) \qquad (43)$$

To start the iteration, we select an initial guess y_0.

Let $y = y(x,0) + ty_t(x,0)$. Then we expand y as a Legendre wavelets,

$$C_{n+1}^T = C_0^T + (C_n^TD_x^2 - b_1C_n^T + b_2G_n^T + F^T)P_t^2 \qquad (44)$$

4. CONVERGENCE ANALYSIS

The series solution of Eq. (12) using LLWM converges towards u(x).

Let U* is the exact solution of Eq. (12), we have

$$U^* = U_0 + \prod[\alpha U_{xx}^*] \qquad (45)$$

$$\text{and } U_{n+1} = U_0 + \prod[\alpha(u_n)_{xx}] \qquad (46)$$

$$U_{n+1} = U_0 + \prod[\alpha(U_n - U^*)_{xx}] \qquad (47)$$

Using the Lipschitz condition,

$$\|g(u_n) - g(u^*)\| \leq \gamma\|U_n - U^*\| \qquad (48)$$

$$\|U_{n+1} - U^*\| \leq \|\prod[\alpha(U_n - U^*)_{xx}]\| \qquad (49)$$

Let $U_{n+1} = C_{n+1}^T \Psi(x,t)$

$$U^* = C^T\Psi(x,t) \qquad (50)$$

and $\xi_{n+1}^T = C_{n+1}^T - C_n^T$

$$\xi_{n+1}^T \leq \xi_n^T\|D_x^2P_t^2\| \qquad (51)$$

By recursion, we get

$$\xi_{n+1}^T \leq \|D_x^2P_t^2\|\xi^0 \qquad (52)$$

where $\lim_{n \to \infty} \| D_x^2 P_t^2 \| = 0$, the series of solution of Eq. (12) using the LLWM converges to $U^*(x)$.

by taking k= k'=1 and M = M', the maximum element of P_t and D_x is ½ and $2\sqrt{(2M-1)(2M-3)}$ respectively.

5. NUMERICAL EXAMPLES

Example: 5. 1 We consider the simple homogeneous wave equation Eq. (12) with boundary conditions

u(0, t) =0, u(0, 1) = 0, t≥ 0

(53) and the initial conditions

$$u(x,0) = u_0 \sin(\pi x), \ u_t(x, 0) = 0 \tag{54}$$

Using Homotopy analysis method (HAM), the exact solution in a closed form is given by

$$u(x,t) = u_0 \cos(\pi at)\sin(\pi x) \tag{55}$$

The LLWM scheme is given by

$$C_{n+1}^T = C_0^T + [C_n^T \ D_x^2] P_t^2 \tag{56}$$

Table 1: Comparison between the exact solution and the LLWM for
$$\alpha = 1, u_0 = 1, t = 0.2, k = 1 \ and \ M = 16.$$

$x/64$	Error in LLWM E_{LLWM}	Error in HAM E_{HAM}
1	-1. 522e-002	-1. 520e-002
3	-6. 623e-002	-6. 621e-002
5	-6. 988e-002	-5. 989e-002
7	-3. 230e-002	-1. 231e-002
9	-3. 052e-002	-2. 0523e-002
11	-7. 993e-002	-4. 993e-002
13	-3. 080e-002	-9. 081e-002

We have utilized the LLWM to solve Eq. (12) with k=1 and M=16. Table. 1 exhibits the comparison of the exact solutions and the LLWM solutions for different values of

x. The numerical results show that the LLWM method is quite reasonable when compare to exact

Example:5. 2 Consider the fractional nonlinear partial differential equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} = -u^2, \ 0 \leq x, t < \ 1, 1 \leq \alpha < 2, \tag{57}$$

$$u(x,0) = u_0(x,0) = 1 + \sin x, \ \frac{\partial}{\partial t} u(x,0) = 0 \tag{58}$$

Using HAM, the exact solution in a closed form is given by

$$u(x,t) = 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}, \dots \right) + \frac{t^\alpha}{\Gamma(\alpha+1)} \left(-1 - 3x - x^2 + \frac{3x^3}{3!} + \frac{x^4}{3} - \frac{3x^5}{5!}, \dots \right)$$

$$+ \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \left(11x + 12x^2 - \frac{11x^3}{3!} - 4x^4, \dots \right) + \dots \tag{59}$$

The proposed LLWM is given by

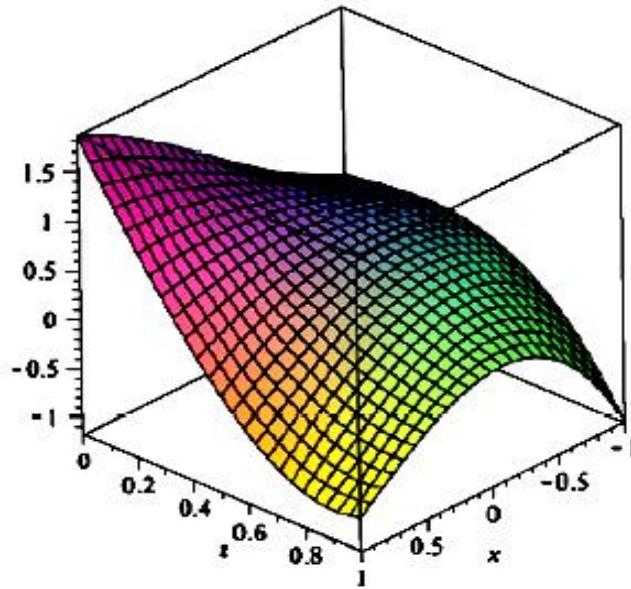$$C_{n+1}^T = C_0^T + (C_n^T D_x^\alpha - (C_n^T)^2) P_t^2 \tag{60}$$

Figure 1: Legendre wavelet solution u(x, t) for Eq. (57) when $\alpha = 1$ and M=3.

Our results can be compared with Alomari et al. [33] results. Good agreement with the exact solution is observed. Fig. 1 shows the numerical solution of Eq. (57).

CONCLUSION

This paper provides an efficient wavelet method is applied for solving a few wave-type and time-fractional wave equations. It offers a state-of-the-art in several active areas of research where numerical methods for solving partial differential equations have proved particularly effective. The proposed schemes are the capability to overcome the difficulty arising in calculating the integral values while dealing with nonlinear problems. This method shows higher efficiency than the traditional methods for solving nonlinear PDEs.

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