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Some Common Fixed Point Theorems for Four Self - Maps Satisfying Generalized (ϕ, ψ) – Weak Contraction in Complete Metric Space

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Abstract

The aim of this manuscript is to establish some common fixed point theorems for four weakly compatible self-mappings $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ in a complete metric space (M, d^+) satisfying the generalized (ϕ, ψ) – weak contraction condition. An example is also given to support our results.

Keywords: fixed point, coincidence point, weakly compatible maps, altering distance function.

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1. INTRODUCTION

The theory of the fixed points is one of the most powerful tools of modern mathematics. It is a beautiful mixture of analysis, topology and geometry. Theorems concerning the existence and properties of fixed points are known as the fixedpoint theorems. Fixed point theorems have been applied in such fields as mathematics, engineering physics, economics, game theory, biology and chemistry etc.

Let X be any non empty set. A map $f : X \to X$ is said to have a fixed point $x \in X$ if $f(x) = x$. Fixed points appear in various branches of mathematics and play a key role in proving some crucial results.

For example: Let $f(x) = x^2$, To find the fixed points, put $f(x) = x$, i.e., $x^2 = x$.

Clearly, 0 and 1 are the fixed points of $f(x) = x^2$.

2. PRELIMANIRIES

Definition 2.1. A coincidence point of a pair of self – maps $\hat{P}, \hat{Q} : M \to M$ is a point $\mu \in M$ for which $P\mu = Q\mu$.

A common fixed point of a pair of self - mappings- $\hat{P}, \hat{Q} : M \to M$ is a point $\mu \in M$ for which $\hat{P}\mu = \hat{Q}\mu = \mu$.

In 1996, Jungck [2] introduced the concept of weakly compatible mappings to study common fixed point theorems:

Definition 2.2. Let (M, d^*) be a metric space. A pair of self – maps $\hat{P}, \hat{Q} : M \to M$ is weakly compatible if they commute at their coincidence points, that is, if there exists $\mu \in M$ such that $\hat{P}\hat{Q}\mu = \hat{Q}\hat{P}\mu$, where μ is coincidence point of P and Q.

Definition 2.3.(Altering distance function)[5] A function $\psi: [0, \infty) \to [0, \infty)$ is called an altering function if it satisfied the following conditions:

a) ψ is monotonic increasing and continuous function.

b) $\psi(t) = 0$ if and only if $t = 0$.

In 2015, Murthy *et al.* proved a common fixed point theorem for two pairs of self-maps in complete metric space with (ϕ, ψ) – weak contraction condition.

The main purpose of this paper is to establish a few common fixed point theorems by generalizing the result of Murthy *et al.* [7] for two pairs of self-maps $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ in complete metric space (M, d^+) with generalized (ϕ, ψ) – weak contraction condition

$$\psi\left(d^+(\hat{P}\mu, \hat{Q}v)\right) \le \psi(\Delta_1(\mu, \nu)) - \phi(\Delta_2(\mu, \nu)),$$

for all μ, v in M with $\mu \neq v$,

where

$$\Delta_1(\mu, v) = \max\Big\{d^+(\hat{S}\mu, \hat{T}v), d^+(\hat{S}\mu, \hat{P}\mu), d^+(\hat{Q}v, \hat{T}v), \tfrac{1}{2}[d^+(\hat{S}\mu, \hat{P}\mu) + d^+(\hat{T}v, \hat{Q}v)],$$
$$d^+(\hat{P}\mu, \hat{S}\mu)\left[\frac{1 + d^+(\hat{S}\mu, \hat{T}v) + d^+(\hat{Q}v, \hat{P}\mu)}{1 + d^+(\hat{P}\mu, \hat{S}\mu) + d^+(\hat{T}v, \hat{Q}v)}\right]\Big\}$$

and

$$\Delta_2(\mu, v) = \min\Big\{d^+(\hat{S}\mu, \hat{T}v), d^+(\hat{S}\mu, \hat{P}\mu), d^+(\hat{Q}v, \hat{T}v), \tfrac{1}{2}[d^+(\hat{S}\mu, \hat{P}\mu) + d^+(\hat{T}v, \hat{Q}v)],$$
$$d^+(\hat{P}\mu, \hat{S}\mu)\left[\frac{1 + d^+(\hat{S}\mu, \hat{T}v) + d^+(\hat{Q}v, \hat{P}\mu)}{1 + d^+(\hat{P}\mu, \hat{S}\mu) + d^+(\hat{T}v, \hat{Q}v)}\right]\Big\}$$

3. MAIN RESULTS

Theorem 3.1. Let (M, d^+) be a complete metric space and let $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ be self-maps on M satisfying the followings:

(3.1) $\hat{P}M \subseteq \hat{T}M, \hat{Q}M \subseteq \hat{S}M$;

(3.2) $(\hat{P},\hat{S})$ and $(\hat{Q},\hat{T})$ are weakly compatible;

(3.3) $\psi\left(d^+\left(\hat{P}\mu,\hat{Q}\upsilon\right)\right) \leq \psi\left(\Delta_1(\mu,\nu)\right) - \phi\left(\Delta_2(\mu,\nu)\right)$,

for all μ,υ in M with $\mu \neq \upsilon$,

where

$$\Delta_1(\mu,\upsilon) = max\Big\{d^+\left(\hat{S}\mu,\hat{T}\upsilon\right), d^+\left(\hat{S}\mu,\hat{P}\mu\right), d^+\left(\hat{Q}\upsilon,\hat{T}\upsilon\right), \frac{1}{2}\left[d^+\left(\hat{S}\mu,\hat{P}\mu\right) + d^+(\hat{T}\upsilon,\hat{Q}\upsilon)\right], d^+(\hat{P}\mu,\hat{S}\mu)\left[\frac{1 + d^+\left(\hat{S}\mu,\hat{T}\upsilon\right) + d^+(\hat{Q}\upsilon,\hat{P}\mu)}{1 + d^+\left(\hat{P}\mu,\hat{S}\mu\right) + d^+(\hat{T}\upsilon,\hat{Q}\upsilon)}\right]\Big\},$$

and

$$\Delta_2(\mu,\upsilon) = min\Big\{d^+\left(\hat{S}\mu,\hat{T}\upsilon\right), d^+\left(\hat{S}\mu,\hat{P}\mu\right), d^+\left(\hat{Q}\upsilon,\hat{T}\upsilon\right), \frac{1}{2}\left[d^+\left(\hat{S}\mu,\hat{P}\mu\right) + d^+(\hat{T}\upsilon,\hat{Q}\upsilon)\right], d^+(\hat{P}\mu,\hat{S}\mu)\left[\frac{1 + d^+\left(\hat{S}\mu,\hat{T}\upsilon\right) + d^+(\hat{Q}\upsilon,\hat{P}\mu)}{1 + d^+\left(\hat{P}\mu,\hat{S}\mu\right) + d^+(\hat{T}\upsilon,\hat{Q}\upsilon)}\right]\Big\},$$

for all u,υ in M.

(3.4) $\phi: [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ which is lower semi – continuous for all

$t > 0$,and ϕ is discontinuous at $t = 0$ with $t(0) = 0$,

(3.5) $\psi: [0,\infty) \to [0,\infty)$ is an altering function.

Then $\hat{P},\hat{Q},\hat{S}$ and $\hat{T}$ have a unique common fixed point in M.

Proof: Let $\mu_0 \in M$ be an arbitrary point of M. From (3.1), since $\hat{P}M \subseteq \hat{T}M$ and $\hat{Q}M \subseteq \hat{S}M$, we can construct two sequences $\{\mu_n\}$ and$\{\upsilon_n\}$ in M as follows:

$$\upsilon_{2n+1} = \hat{P}\mu_{2n} = \hat{T}\mu_{2n+1}, \ \upsilon_{2n+2} = \hat{Q}\mu_{2n+1} = \hat{S}\mu_{2n+2}, \tag{3.6}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Now, we define $d^+{}_{\mathrm{n}} = d^+(\upsilon_{\mathrm{n}},\upsilon_{\mathrm{n+1}})$ for each $n \in \mathbb{N}$.

On putting, $\mu = \mu_{2n}$ and $\upsilon = \mu_{2n+1}$ in (3.3) and using (3.6), we get

$$\psi\left(d^+\left(\hat{P}\mu_{2n},\hat{Q}\mu_{2n+1}\right)\right) = \psi\left(d^+(\upsilon_{2n+1},\upsilon_{2n+2})\right)$$

$$\psi d^+{}_{\mathrm{2n+1}} \leq \psi\left(\Delta_1(\mu_{2n},\mu_{2n+1})\right) - \phi\left(\Delta_2(\mu_{2n},\mu_{2n+1})\right), \tag{3.7}$$

where

$\Delta_1(\mu_{2n},\mu_{2n+1}) = max\Big\{d^+\left(\hat{S}\mu_{2n},\hat{T}\mu_{2n+1}\right), d^+\left(\hat{S}\mu_{2n},\hat{P}\mu_{2n}\right), d^+\left(\hat{T}\mu_{2n+1},\hat{Q}\mu_{2n+1}\right),$

$\frac{1}{2}\left[d^+\left(\hat{S}\mu_{2n},\hat{P}\mu_{2n}\right) + d^+(\hat{T}\mu_{2n+1},\hat{Q}\mu_{2n+1})\right],$

$d^+(\hat{P}\mu_{2n}, \hat{S}\mu_{2n})\left[\frac{1 + d^+(\hat{S}\mu_{2n}, \hat{T}\mu_{2n+1}) + d^+(\hat{Q}\mu_{2n+1}, \hat{P}\mu_{2n})}{1 + d^+(\hat{P}\mu_{2n}, \hat{S}\mu_{2n}) + d^+(\hat{T}\mu_{2n+1}, \hat{Q}\mu_{2n+1})}\right]\Big\}.$

$= max\{d^+(\upsilon_{2n,}\upsilon_{2n+1}), d^+(\upsilon_{2n,}\upsilon_{2n+1}), d^+(\upsilon_{2n+1,}\upsilon_{2n+2}),$

$\frac{1}{2}[d^+(\upsilon_{2n,}\upsilon_{2n+1}) + d^+(\upsilon_{2n+1,}\upsilon_{2n+2})],$

$d^+(\upsilon_{2n+1,}\upsilon_{2n})\frac{1 + d^+(\upsilon_{2n,}\upsilon_{2n+1}) + d^+(\upsilon_{2n+2,}\upsilon_{2n+1})}{1 + d^+(\upsilon_{2n+1,}\upsilon_{2n}) + d^+(\upsilon_{2n+1,}\upsilon_{2n+2})}\Big\}.$

$= \ max \left\{d^+{}_{2n}, \ d^+{}_{2n}, \quad d^+{}_{2n+1}, \quad \frac{1}{2}[d^+{}_{2n} + d^+{}_{2n+1}], d^+{}_{2n}\frac{1 + d^+{}_{2n} + d^+{}_{2n+1}}{1 + d^+{}_{2n} + d^+{}_{2n+1}}\right\}.$

$= \ max\ \{d^+{}_{2n}, d^+{}_{2n+1}\}.$

If $d^+{}_{2n} < d^+{}_{2n+1}$,

$\Delta_1(\mu_{2n}, \mu_{2n+1}) = d^+{}_{2n+1}.$

And

$\Delta_2(\mu_{2n}, \mu_{2n+1}) = \min\{d^+{}_{2n}, d^+{}_{2n+1}\}.$

$\Delta_2(\mu_{2n}, \mu_{2n+1}) = d^+{}_{2n}$

$\psi\left(d^+(\hat{P}\mu_{2n}, \hat{Q}\mu_{2n+1})\right) = \psi(d^+(\upsilon_{2n+1}, \upsilon_{2n+2}))$

$$\psi(d^+{}_{2n+1}) \ \le \psi(\Delta_1(\mu_{2n}, \mu_{2n+1})) - \phi(\Delta_2(\mu_{2n}, \mu_{2n+1})), \tag{3.8}$$

$$< \psi(d^+{}_{2n+1}),$$

a contradiction since $\phi(t) > 0$ when $t > 0$.

Hence, $d^+{}_{2n+1} < d^+{}_{2n} = \Delta_1(\mu_{2n}, \mu_{2n+1})$ for all n in $\mathbb{N}$.

Similarly,

$$d^+{}_{2n} < d^+{}_{2n-1} = \Delta_1(\mu_{2n}, \mu_{2n-1}) \text{ for all } n \text{ in } \mathbb{N},$$

which implies that

$$d^+{}_{n+1} < d^+{}_{n,} d^+{}_{2n} = \Delta_1(\mu_{2n}, \mu_{2n+1}), d^+{}_{2n-1} = \Delta_1(\mu_{2n}, \mu_{2n-1}),$$

which implies that $\{\ d^+(\upsilon_{2n}, \upsilon_{2n+1})\}$ is monotonic decreasing sequence bounded below and there exists a constant k such that,

$$\lim_{n\to\infty} d^+(\upsilon_{2n}, \upsilon_{2n+1}) = k \ge 0. \tag{3.9}$$

Now, by equation (3.8) and using (3.9), we get

$$\lim_{n\to\infty} \psi(d^+(\upsilon_{2n+1}, \upsilon_{2n+2})) \ \le \lim_{n\to\infty} \psi(\Delta_1(\mu_{2n}, \mu_{2n+1})) - \phi(\Delta_2(\mu_{2n}, \mu_{2n+1})),$$

$$\le \lim_{n\to\infty} \psi(d^+(\upsilon_{2n}, \upsilon_{2n+1})) - \lim_{n\to\infty} \phi(0), \tag{3.10}$$

using the discontinuity of $\phi(t)$ at $t = 0$ and equation (3.9), we observe that right hand side of equation (3.10) is non zero. Therefore, we get

$$\psi(k) < \psi(k),$$

a contradiction. Hence, we have

$$\lim_{n\to\infty} d^+(\upsilon_{2n}, \upsilon_{2n+1}) = 0. \tag{3.11}$$

Now, we prove that $\{\upsilon_n\}$is a Cauchy sequence. For this, it is sufficient to show that $\{\upsilon_{2n}\}$is a Cauchy sequence. On contrary basis, let, if possible $\{\upsilon_{2n}\}$ is not a Cauchy sequence. Then, there exists $\epsilon > 0$ and $n,\ m > 0$ with $2n(\alpha) > 2m(\alpha) > 2\alpha$ satisfying

$$d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2n(\alpha)}\big) \geq \epsilon, \text{ for all } \alpha \in \mathbb{N} \tag{3.12}$$

where $2n(\alpha)$ is the least integer exceeding $2m(\alpha)$ satisfying (3.12). It follows that

$$d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2n(\alpha)-1}\big) < \epsilon, \text{ for all } \alpha \in \mathbb{N}. \tag{3.13}$$

Now, using (3.12) and triangular inequality, we obtain the following:

$$\epsilon \leq d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2n(\alpha)}\big),$$

$$\leq d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2n(\alpha)-1}\big) + d^+\big(\upsilon_{2n(\alpha)-1}, \upsilon_{2n(\alpha)}\big), \tag{3.14}$$

taking limit as $\alpha \to \infty$, we get

$$\lim_{\alpha\to\infty} d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2n(\alpha)}\big) = \epsilon \tag{3.15}$$

Again for all α, we have

$$d^+\big(\upsilon_{2m(\alpha)-1}, \upsilon_{2n(\alpha)-1}\big) \leq d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2m(\alpha)-1}\big) + d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2n(\alpha)}\big)$$
$$+ d^+\big(\upsilon_{2n(\alpha)-1}, \upsilon_{2n(\alpha)}\big),$$
$$d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2n(\alpha)}\big) \leq d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2m(\alpha)-1}\big) + d^+\big(\upsilon_{2m(\alpha)-1}, \upsilon_{2n(\alpha)-1}\big)$$
$$+ d^+\big(\upsilon_{2n(\alpha)-1}, \upsilon_{2n(\alpha)}\big).$$

On letting $\alpha \to \infty$ and using (3.11) – (3.15), we get

$$\lim_{\alpha\to\infty} d^+\big(\upsilon_{2m(\alpha)-1}, \upsilon_{2n(\alpha)-1}\big) = \epsilon. \tag{3.16}$$

Again for all α, we have

$$d^+\big(\upsilon_{2m(\alpha)-1}, \upsilon_{2n(\alpha)}\big) \leq d^+\big(\upsilon_{2m(\alpha)-1}, \upsilon_{2m(\alpha)}\big) + d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2n(\alpha)}\big),$$
$$d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2n(\alpha)}\big) \leq d^+\big(\upsilon_{2m(\alpha)}, \upsilon_{2m(\alpha)-1}\big) + d^+\big(\upsilon_{2m(\alpha)-1}, \upsilon_{2n(\alpha)}\big).$$

On letting $\alpha \to \infty$ and using (3.11) – (3.16), we get

$$\lim_{\alpha\to\infty} d^+\big(\upsilon_{2m(\alpha)-1}, \upsilon_{2n(\alpha)}\big) = \epsilon. \tag{3.17}$$

Again for all α, we have

$$d^+(\upsilon_{2n(\alpha)-1}, \upsilon_{2m(\alpha)}) \leq d^+(\upsilon_{2n(\alpha)-1}, \upsilon_{2n(\alpha)}) + d^+(\upsilon_{2n(\alpha)}, \upsilon_{2m(\alpha)}),$$

$$d^+(\upsilon_{2n(\alpha)}, \upsilon_{2m(\alpha)}) \leq d^+(\upsilon_{2n(\alpha)}, \upsilon_{2n(\alpha)-1}) + d^+(\upsilon_{2n(\alpha)-1}, \upsilon_{2m(\alpha)}).$$

On letting $\alpha \to \infty$ and using (3.11) – (3.17), we get

$$\lim_{\alpha\to\infty} d^+(\upsilon_{2n(\alpha)-1}, \upsilon_{2m(\alpha)}) = \epsilon. \tag{3.18}$$

Now, on putting $\mu = \mu_{2m(\alpha)-1}$ and $\upsilon = \mu_{2n(\alpha)-1}$ in (3.3), we obtain that

$$\begin{aligned}\psi\left(d^+(\hat{P}\mu_{2m(\alpha)-1}, \hat{Q}\mu_{2n(\alpha)-1})\right) &= \psi\left(d^+(\upsilon_{2m(\alpha)}, \upsilon_{2n(\alpha)})\right)\\ &\leq \psi\left(\Delta_1(\mu_{2m(\alpha)-1}, \mu_{2n(\alpha)-1})\right) - \phi\left(\Delta_2(\mu_{2m(\alpha)-1}, \mu_{2n(\alpha)-1})\right),\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}\Delta_1(\mu_{2m(\alpha)-1}, \mu_{2n(\alpha)-1}) = max\{&d^+(\hat{S}\mu_{2m(\alpha)-1}, \hat{T}\mu_{2n(\alpha)-1}), d^+(\hat{S}\mu_{2m(\alpha)-1}, \hat{P}\mu_{2m(\alpha)-1}),\\ &d^+(\hat{T}\mu_{2n(\alpha)-1}, \hat{Q}\mu_{2n(\alpha)-1}),\\ &\frac{1}{2}[d^+(\hat{S}\mu_{2m(\alpha)-1}, \hat{P}\mu_{2m(\alpha)-1}) + d^+(\hat{Q}\mu_{2n(\alpha)-1}, \hat{T}\mu_{2n(\alpha)-1})],\\ &d^+(\hat{S}\mu_{2m(\alpha)-1}, \hat{P}\mu_{2m(\alpha)-1})\left[\frac{1 + d^+(\hat{S}\mu_{2m(\alpha)-1}, \hat{T}\mu_{2n(\alpha)-1}) + d^+(\hat{Q}\mu_{2n(\alpha)-1}, \hat{P}\mu_{2m(\alpha)-1})}{1 + d^+(\hat{S}\mu_{2m(\alpha)-1}, \hat{P}\mu_{2m(\alpha)-1}) + d^+(\hat{T}\mu_{2n(\alpha)-1}, \hat{Q}\mu_{2n(\alpha)-1})}\right]\Bigg\}.\end{aligned}$$

$$\begin{aligned}= max\{&d^+(\upsilon_{2m(\alpha)-1}, \upsilon_{2n(\alpha)-1}), d^+(\upsilon_{2m(\alpha)-1}, \upsilon_{2m(\alpha)}),\\ &d^+(\upsilon_{2m(\alpha)-1}, \upsilon_{2m(\alpha)})\frac{1 + d^+(\upsilon_{2m(\alpha)-1}, \upsilon_{2n(\alpha)-1}) + d^+(\upsilon_{2n(\alpha)}, \upsilon_{2m(\alpha)})}{1 + d^+(\upsilon_{2m(\alpha)-1}, \upsilon_{2m(\alpha)}) + d^+(\upsilon_{2n(\alpha)-1}, \upsilon_{2n(\alpha)})}\Bigg\}.\end{aligned}$$

$$= max\left\{\epsilon, 0, 0, \frac{1}{2}[0+0], 0\right\}.$$

$= \epsilon$ as $\alpha \to \infty$.

And

$$\Delta_2(\mu_{2m(\alpha)-1}, \mu_{2n(\alpha)-1}) = 0, \text{as } \alpha \to \infty.$$

Now, from (3.19), we have

$$\begin{aligned}\lim_{\alpha\to\infty}\psi\left(d^+(\upsilon_{2m(\alpha)}, \upsilon_{2n(\alpha)})\right) \leq \lim_{\alpha\to\infty}&\psi\left(\Delta_1(\mu_{2m(\alpha)-1}, \mu_{2n(\alpha)-1})\right)\\ &-\lim_{\alpha\to\infty}\phi\left(\Delta_2(\mu_{2m(\alpha)-1}, \mu_{2n(\alpha)-1})\right),\end{aligned}$$

$$\psi(\epsilon) \leq \psi(\epsilon) - \lim_{\alpha\to\infty}\phi\left(\Delta_2(\mu_{2m(\alpha)-1}, \mu_{2n(\alpha)-1})\right).$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$, we observe that the last term on the right-hand side is non-zero. Thus we arrive at a contradiction.

Hence $\{v_n\}$ is a Cauchy sequence.

Since (M, d^+) is complete metric space, therefore, the Cauchy sequence $\{v_n\}$ is a convergent sequence and it converges to a point z_o(say) in M.

Consequently, the subsequences also converge to z_o in M:

$\hat{P}\mu_{2n} \to z_o$, $\quad \hat{T}\mu_{2n+1} \to z_{o,} \hat{Q}\mu_{2n+1} \to z_o \quad$ and $\quad \hat{S}\mu_{2n} \to z_o$.

Now, we shall prove that z_o is the common fixed point of $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$.

Since$\hat{P}M \subseteq \hat{T}M, \hat{Q}M \subseteq \hat{S}M$, there exists $v_o \in M$, such that $z_o = \hat{S}v_o$ and $z_o = \hat{T}v_o$.

Let $d^+(z_o, \hat{P}v_o) \neq 0$. On putting, $\mu = v_o$ and $v = \mu_{2n+1}$ in (3.3), we have

$$\psi\left(d^+\left(\hat{P}v_o, \hat{Q}\mu_{2n+1}\right)\right) \leq \psi\left(\Delta_1(v_o, \mu_{2n+1})\right) - \phi\left(\Delta_2(v_o, \mu_{2n+1})\right), \tag{3.20}$$

where

$$\Delta_1(v_o, \mu_{2n+1}) = max\{d^+(\hat{S}v_o, \hat{T}\mu_{2n+1}), d^+(\hat{S}v_o, \hat{P}v_o), d^+(\hat{T}\mu_{2n+1}, \hat{Q}\mu_{2n+1}),$$
$$\frac{1}{2}[d^+(\hat{S}v_o, \hat{P}v_o) + d^+(\hat{T}\mu_{2n+1}, \hat{Q}\mu_{2n+1})],$$
$$d^+(\hat{S}v_o, \hat{P}v_o)\left[\frac{1 + d^+(\hat{S}v_o, \hat{T}\mu_{2n+1}) + d^+(\hat{Q}\mu_{2n+1}, \hat{P}v_o)}{1 + d^+(\hat{S}v_o, \hat{P}v_o) + d^+(\hat{T}\mu_{2n+1}, \hat{Q}\mu_{2n+1})}\right]\}.$$

Now, taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \Delta_1(v_o, \mu_{2n+1}) = max\left\{d^+(z_{o,} z_o), d^+(z_o, \hat{P}v_o), d^*(z_o, z_o), \frac{1}{2}[d^+(\hat{P}v_o, z_o) + d^+(z_o, z_o)],\right.$$
$$\left. d^+(\hat{P}v_o, z_o)\frac{1 + d^+(z_o, z_o) + d^+(\hat{P}v_o, z_o)}{1 + d^+(\hat{P}v_o, z_o) + d^+(z_o, z_o)}\right\}$$
$$= max\left\{0, d^+(z_o, \hat{P}v_o), 0, \frac{1}{2}d^+(z_o, \hat{P}v_o), d^+(z_o, \hat{P}v_o)\right\}.$$
$$= d^+(z_o, \hat{P}v_o).$$

And

$$\lim_{n\to\infty} \Delta_2(v_o, \mu_{2n+1}) = 0.$$

Now, from (3.20), we obtain

$$\lim_{n\to\infty} \psi\left(d^+\left(\hat{P}v_o, \hat{Q}\mu_{2n+1}\right)\right) \leq \lim_{n\to\infty} \psi\left(\Delta_1(v_o, \mu_{2n+1})\right) - \lim_{n\to\infty} \phi\left(\Delta_2(v_o, \mu_{2n+1})\right),$$
$$\psi\left(d^+\left(\hat{P}v_o, z_o\right)\right) \leq \psi\left(d^+\left(z_o, \hat{P}v_o\right)\right) - \lim_{n\to\infty} \phi\left(\Delta_2(v_o, \mu_{2n+1})\right),$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$. We observe that the last term on the right-hand side of above inequality is non-zero. Therefore, we obtain

$$\psi\left(d^+\left(\hat{P}v_o, z_o\right)\right) < \psi\left(d^+\left(z_o, \hat{P}v_o\right)\right).$$

Hence we reach at a contradiction with ψ function.

Therefore, $d^+(\hat{P}v_o, z_o) = 0$, this implies that $\hat{P}v_o = z_o = \hat{S}v_o$.

Since $(\hat{P}, \hat{S})$ is weakly compatible pair of maps, so it commutes at their coincidence point v_o.

$$\text{i.e., } \hat{P}\hat{S}v_o = \hat{S}\hat{P}v_o \text{ this implies} \hat{P}z_o = \hat{S}z_o.$$

Now, we shall prove that $\hat{P}z_o = \hat{S}z_o = z_o$.

On putting $\mu = z_o$ and $\upsilon = \mu_{2n+1}$ in (3.3), we have

$$\psi\left(d^+\left(\hat{P}z_o, \hat{Q}\mu_{2n+1}\right)\right) \leq \psi\left(\Delta_1(z_o, \mu_{2n+1})\right) - \phi\left(\Delta_2(z_o, \mu_{2n+1})\right), \tag{3.21}$$

where

$$\Delta_1(z_o, \mu_{2n+1}) = max\{d^+\left(\hat{S}z_o, \hat{T}\mu_{2n+1}\right), d^+\left(\hat{S}z_o, \hat{P}z_o\right), d^+\left(\hat{T}\mu_{2n+1}, \hat{Q}\mu_{2n+1}\right),$$

$$\frac{1}{2}\left[d^+\left(\hat{S}z_o, \hat{P}z_o\right) + d^+(\hat{T}\mu_{2n+1}, \hat{Q}\mu_{2n+1})\right],$$

$$d^+(\hat{S}z_o, \hat{P}z_o)\left[\frac{1 + d^+\left(\hat{S}z_o, \hat{T}\mu_{2n+1}\right) + d^+(\hat{Q}\mu_{2n+1}, \hat{P}z_o)}{1 + d^+\left(\hat{S}z_o, \hat{P}z_o\right) + d^+(\hat{T}\mu_{2n+1}, \hat{Q}\mu_{2n+1})}\right]\Bigg\}.$$

Now, taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \Delta_1(z_o, \mu_{2n+1}) = max\left\{d^+\left(\hat{S}z_o, z_o\right), d^+\left(\hat{S}z_o, \hat{S}z_o\right), d^+(z_o, z_o), \frac{1}{2}\left[d^+\left(\hat{S}z_o, \hat{S}z_o\right) + d^+(z_o, z_o)\right],\right.$$
$$\left. d^+(\hat{S}z_o, \hat{P}v_o)\,\frac{1+d^+(\hat{S}z_o,z_o)+d^+(\hat{P}z_o,z_o)}{1+d^+(\hat{S}z_o,\hat{P}z_o)+d^+(z_o,z_o)}\right\}.$$

$$= \max\left\{d^+\left(\hat{S}z_o, z_o\right), 0, 0, 0, 0\right\} = d^+\left(\hat{S}z_o, z_o\right).$$

And

$$\lim_{n\to\infty} \Delta_2(z_o, \mu_{2n+1}) = 0.$$

From (3.21), we obtain

$$\lim_{n\to\infty} \psi\left(d^+\left(\hat{P}z_o, \hat{Q}\mu_{2n+1}\right)\right) \leq \lim_{n\to\infty} \psi\left(\Delta_1(z_o, \mu_{2n+1})\right) - \lim_{n\to\infty} \phi\left(\Delta_2(z_o, \mu_{2n+1})\right),$$

$$\psi\left(d^+\left(\hat{S}z_o, z_o\right)\right) \leq \psi\left(d^+\left(\hat{S}z_o, z_o\right)\right) - \lim_{n\to\infty} \phi\left(\Delta_2(v_o, \mu_{2n+1})\right),$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$. We observe that the last term on the right-hand side of above inequality is non-zero. Therefore, we obtain

$$\psi\left(d^+\left(\hat{S}z_o, z_o\right)\right) < \psi\left(d^+\left(\hat{S}z_o, z_o\right)\right).$$

Hence we reach at a contradiction with ψ function.

Therefore $d^+\left(\hat{S}z_o, z_o\right) = 0$, this implies that $\hat{P}z_o = z_o = \hat{S}z_o$. Similarly, we can show that $\hat{T}z_o = z_o = \hat{Q}z_o$.

Hence $\hat{P}z_o = \hat{S}z_o = \hat{Q}z_o = \hat{T}z_o = z_o$.

This implies that z_o is the common fixed point of $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$.

Now, we shall show that z_o is the unique common fixed point of $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$.

Let z_1 be the another fixed point of $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ such that $z_o \neq z_1$.

On putting $\mu = z_o$ and $\upsilon = z_1$ in (3.3), we have

$$\psi\left(d^{+}\left(\hat{P}z_o, \hat{Q}z_1\right)\right) \leq \psi\left(\Delta_1(z_o, z_1)\right) - \phi\left(\Delta_2(z_o, z_1)\right), \tag{3.22}$$

$$\psi\left(d^{+}(z_o, z_1)\right) \leq \psi\left(d^{+}(z_o, z_1)\right) - \phi\left(d^{+}(z_o, z_1)\right),$$

A contradiction. Hence $d^{+}(z_o, z_1) = 0$, this implies, $z_o = z_1$. Hence $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ have unique fixed point in M. This completes the proof of the theorem.

Note: When we take $\hat{T} = \hat{S} = I$ identity map, we get the following theorem.

Theorem 3.2. Let (M, d^{+}) be a complete metric space and let $\hat{P}, \hat{Q}$ be self-maps on M satisfying the followings:

(3.23) $\psi\left(d^{+}\left(\hat{P}\mu, \hat{Q}\upsilon\right)\right) \leq \psi\left(\Delta_1(\mu, \nu)\right) - \phi\left(\Delta_2(\mu, \nu)\right)$,

for all μ, υ in M with $\mu \neq \upsilon$,

where

$\Delta_1(\mu, \upsilon) = \max\{d^{+}(\mu, \upsilon), d^{+}\left(\mu, \hat{P}\mu\right), d^{+}\left(\hat{Q}\upsilon, \upsilon\right), \frac{1}{2}\left[d^{+}\left(\mu, \hat{P}\mu\right) + d^{+}(\upsilon, \hat{Q}\upsilon)\right]$,

$d^{+}(\hat{P}\mu, \mu)\left[\dfrac{1 + d^{+}(\mu, \upsilon) + d^{+}(\hat{Q}\upsilon, \hat{P}\mu)}{1 + d^{+}\left(\hat{P}\mu, \mu\right) + d^{+}(\upsilon, \hat{Q}\upsilon)}\right]\}$

and

$\Delta_2(\mu, \upsilon) = \min\{d^{+}(\mu, \upsilon), d^{+}\left(\mu, \hat{P}\mu\right), d^{+}\left(\hat{Q}\upsilon, \upsilon\right), \frac{1}{2}\left[d^{+}\left(\mu, \hat{P}\mu\right) + d^{+}(\upsilon, \hat{Q}\upsilon)\right]$,

$d^{+}(\hat{P}\mu, \mu)\left[\dfrac{1 + d^{+}(\mu, \upsilon) + d^{+}(\hat{Q}\upsilon, \hat{P}\mu)}{1 + d^{+}\left(\hat{P}\mu, \mu\right) + d^{+}(\upsilon, \hat{Q}\upsilon)}\right]\}$.

for all u, υ in M.

(3.24) $\phi: [0, \infty) \to [0, \infty)$ is such that $\phi(t) > 0$ which is lower semi – continuous for all

$t > 0$, and ϕ is discontinuous at $t = 0$ with $t(0) = 0$,

(3.25) $\psi: [0, \infty) \to [0, \infty)$ is an altering distance function.

Then $\hat{P}, \hat{Q}$ have a unique common fixed point in M.

Note: When we take $\psi(t) = t$ in Theorem 3.1 and Theorem 3.2, we get the following corrolaries.

Corollary 3.3. Let (M, d^+) be a complete metric space and let $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ be self-maps on M satisfying the followings:

(3.26) $\hat{P}M \subseteq \hat{T}M, \hat{Q}M \subseteq \hat{S}M$;

(3.27) $(\hat{P}, \hat{S})$ and $(\hat{Q}, \hat{T})$ are weakly compatible;

(3.28) $d^+(\hat{P}\mu, \hat{Q}v) \leq \Delta_1(\mu, \nu) - \phi\Delta_2(\mu, \nu)$,

for all μ, v in M with $\mu \neq v$,

where

$$\Delta_1(\mu, v) = max\Big\{d^+(\hat{S}\mu, \hat{T}v), d^+(\hat{S}\mu, \hat{P}\mu), d^+(\hat{Q}v, \hat{T}v), \frac{1}{2}\left[d^+(\hat{S}\mu, \hat{P}\mu) + d^+(\hat{T}v, \hat{Q}v)\right], d^+(\hat{P}\mu, \hat{S}\mu)\left[\frac{1 + d^+(\hat{S}\mu, \hat{T}v) + d^+(\hat{Q}v, \hat{P}\mu)}{1 + d^+(\hat{P}\mu, \hat{S}\mu) + d^+(\hat{T}v, \hat{Q}v)}\right]\Big\}$$

and

$$\Delta_2(\mu, v) = min\Big\{d^+(\hat{S}\mu, \hat{T}v), d^+(\hat{S}\mu, \hat{P}\mu), d^+(\hat{Q}v, \hat{T}v), \frac{1}{2}\left[d^+(\hat{S}\mu, \hat{P}\mu) + d^+(\hat{T}v, \hat{Q}v)\right], d^+(\hat{P}\mu, \hat{S}\mu)\left[\frac{1 + d^+(\hat{S}\mu, \hat{T}v) + d^+(\hat{Q}v, \hat{P}\mu)}{1 + d^+(\hat{P}\mu, \hat{S}\mu) + d^+(\hat{T}v, \hat{Q}v)}\right]\Big\}.$$

for all u, v in M.

(3.29) $\phi: [0, \infty) \to [0, \infty)$ is such that $\phi(t) > 0$ which is lower semi – continuous for all

$t > 0$, and ϕ is discontinuous at $t = 0$ with $t(0) = 0$,

Then $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ have a unique common fixed point in M.

Corollary 3.4 Let (M, d^+) be a complete metric space and let $\hat{P}, \hat{Q}$ be self-maps on M satisfying the followings:

(3.30) $d^+(\hat{P}\mu, \hat{Q}v) \leq \Delta_1(\mu, \nu) - \phi\Delta_2(\mu, \nu)$,

for all μ, v in M with $\mu \neq v$,

where

$$\Delta_1(\mu, v) = max\Big\{d^+(\mu, v), d^+(\mu, \hat{P}\mu), d^+(\hat{Q}v, v), \frac{1}{2}\left[d^+(\mu, \hat{P}\mu) + d^+(v, \hat{Q}v)\right], d^+(\hat{P}\mu, \mu)\left[\frac{1 + d^+(\mu, v) + d^+(\hat{Q}v, \hat{P}\mu)}{1 + d^+(\hat{P}\mu, \mu) + d^+(v, \hat{Q}v)}\right]\Big\}$$

and

$$\Delta_2(\mu, v) = min\Big\{d^+(\mu, v), d^+(\mu, \hat{P}\mu), d^+(\hat{Q}v, v), \frac{1}{2}\left[d^+(\mu, \hat{P}\mu) + d^+(v, \hat{Q}v)\right], d^+(\hat{P}\mu, \mu)\left[\frac{1 + d^+(\mu, v) + d^+(\hat{Q}v, \hat{P}\mu)}{1 + d^+(\hat{P}\mu, \mu) + d^+(v, \hat{Q}v)}\right]\Big\}.$$

for all u, v in M.

(3.31) $\phi: [0,\infty) \to [0,\infty)$ is such that $\phi(t) > 0$ which is lower semi – continuous for all

$t > 0$, and ϕ is discontinuous at $t = 0$ with $t(0) = 0$,

Then $\hat{P}, \hat{Q}$ have a unique common fixed point in M.

Example 3.4. Let$\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$be self - mappings on $M. M = [0,1]$ be endowed with the Euclidean metric $d^{+}(\mu, v) = |\mu - v|$ for all μ, v in M. Let $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$are defined by

$$\hat{Q}\mu = \begin{cases} 0 & if \quad \mu = 0 \\ \dfrac{\mu}{4}, & if \quad \mu > 0, \end{cases}$$

$$\hat{P}\mu = \begin{cases} 0 & if \quad \mu = 0 \\ \dfrac{\mu}{8}, & if \quad \mu > 0, \end{cases}$$

$$\hat{T}\mu = \begin{cases} 0 & if \quad \mu = 0 \\ \dfrac{\mu}{2}, & if \quad \mu > 0, \end{cases}$$

$$\hat{S}\mu = \begin{cases} 0 & if \quad \mu = 0 \\ \mu, & if \quad \mu > 0 \end{cases}$$

$$\psi(t) = \frac{t}{2}, \;\; \phi(t) = \frac{t}{4}, \text{ for all} t \text{ in } \mathbb{R}.$$

Clearly, $\hat{P}M = \left[0, \frac{1}{8}\right] \subseteq \left[0, \frac{1}{2}\right] = \hat{T}M \; and \; \hat{Q}M \left[0, \frac{1}{4}\right] \subseteq [0,1] = \hat{S}M,$ implies that (3.1) is satisfied.

Since $\hat{P}\hat{S}(0) = \hat{S}\hat{P}(0) = 0$, implies that the pair $(\hat{P}, \hat{S})$ is weakly compatible and

$\hat{Q}\hat{T}(0) = \hat{T}\hat{Q}(0) = 0$, implies that the pair $(\hat{Q}, \hat{T})$ is weakly compatible.

Now, we check condition (3.3) for the following cases:

Case 1. If $\mu = 0$ and $\nu = 0$.

$$\psi\left(d^{+}(\hat{P}\mu, \hat{Q}v)\right) = \psi\left(|\hat{P}\mu - \hat{Q}v|\right) = \psi(0) = 0.$$

Also

$$\psi(\Delta_1(\mu, v)) = \psi(0) = 0,$$

$$\phi(\Delta_2(\mu, v)) = \phi(0) = 0.$$

Hence

$$\psi\left(d^{+}\left(\hat{P}\mu,\hat{Q}\nu\right)\right)=\psi\left(\Delta_1(\mu,\nu)\right)-\phi\left(\Delta_2(\mu,\nu)\right).$$

Clearly, inequality (3.4) holds.

Case 2. If $\mu=0,\nu\neq 0$.

$$\psi\left(d^{+}\left(\hat{P}\mu,\hat{Q}\nu\right)\right)=\psi\left(\left|\hat{P}\mu-\hat{Q}\nu\right|\right)=\psi\left(\left|0-\frac{\nu}{4}\right|\right)=\frac{\nu}{8}.$$

And

$$\Delta_1(\mu,\nu)=max\left\{d^{+}\left(\hat{S}\mu,\hat{T}\upsilon\right),d^{+}\left(\hat{S}\mu,\hat{P}\mu\right),d^{+}\left(\hat{Q}\upsilon,\hat{T}\upsilon\right),\frac{1}{2}\left[d^{+}\left(\hat{S}\mu,\hat{P}\mu\right)+d^{+}(\hat{T}\upsilon,\hat{Q}\upsilon)\right],\right.$$
$$\left.d^{+}(\hat{P}\mu,\hat{S}\mu)\left[\frac{1+d^{+}\left(\hat{S}\mu,\hat{T}\upsilon\right)+d^{+}(\hat{Q}\upsilon,\hat{P}\mu)}{1+d^{+}\left(\hat{P}\mu,\hat{S}\mu\right)+d^{+}(\hat{T}\upsilon,\hat{Q}\upsilon)}\right]\right\}$$

$$=\max\{\left|\hat{S}\mu-\hat{T}\nu\right|,\left|\hat{S}\mu-\hat{P}\mu\right|,\left|\hat{Q}\nu-\hat{T}\nu\right|,\frac{1}{2}\left[\left|\hat{S}\mu-\hat{P}\mu\right|+\left|\hat{T}\nu-\hat{Q}\nu\right|\right],$$
$$\left|\hat{P}\mu-\hat{S}\mu\right|\left[\frac{1+\left|\hat{S}\mu-\hat{T}\nu\right|+\left|\hat{Q}\nu-\hat{P}\mu\right|}{1+\left|\hat{P}\mu-\hat{S}\mu\right|+\left|\hat{T}\nu-\hat{Q}\nu\right|}\right]\}$$
$$=\max\left\{\frac{\nu}{2},0,\frac{\nu}{4},\frac{\nu}{8},0\right\}.$$
$$=\frac{\nu}{2}.$$

And

$$\Delta_2(\mu,\upsilon)=0.$$

Hence

$$\psi\left(\Delta_1(\mu,\nu)\right)-\phi\left(\Delta_2(\mu,\nu)\right)=\psi\left(\frac{\nu}{2}\right)-0=\frac{\nu}{4}>\frac{\nu}{8}=\psi\left(d^{+}\left(\hat{P}\mu,\hat{Q}\nu\right)\right).$$

Hence inequality (3.3) satisfied

Case 3. If $\mu\neq 0,\nu=0$.

$$\psi\left(d^{+}\left(\hat{P}\mu,\hat{Q}\nu\right)\right)=\psi\left(\left|\hat{P}\mu-\hat{Q}\nu\right|\right)=\psi\left(\left|\frac{\mu}{8}-0\right|\right)=\frac{\mu}{8}.$$

And

$$\Delta_1(\mu,\nu)=max\left\{d^{+}\left(\hat{S}\mu,\hat{T}\upsilon\right),d^{+}\left(\hat{S}\mu,\hat{P}\mu\right),d^{+}\left(\hat{Q}\upsilon,\hat{T}\upsilon\right),\frac{1}{2}\left[d^{+}\left(\hat{S}\mu,\hat{P}\mu\right)+d^{+}(\hat{T}\upsilon,\hat{Q}\upsilon)\right],\right.$$
$$\left.d^{+}(\hat{P}\mu,\hat{S}\mu)\left[\frac{1+d^{+}\left(\hat{S}\mu,\hat{T}\upsilon\right)+d^{+}(\hat{Q}\upsilon,\hat{P}\mu)}{1+d^{+}\left(\hat{P}\mu,\hat{S}\mu\right)+d^{+}(\hat{T}\upsilon,\hat{Q}\upsilon)}\right]\right\}$$

$$=\max\{\left|\hat{S}\mu-\hat{T}\nu\right|,\left|\hat{S}\mu-\hat{P}\mu\right|,\left|\hat{Q}\nu-\hat{T}\nu\right|,\frac{1}{2}\left[\left|\hat{S}\mu-\hat{P}\mu\right|+\left|\hat{T}\nu-\hat{Q}\nu\right|\right],$$

$$\left|\hat{P}\mu - \hat{S}\mu\right| \left[\frac{1+\left|\hat{S}\mu - \hat{T}\nu\right| + \left|\hat{Q}\nu - \hat{P}\mu\right|}{1+\left|\hat{P}\mu - \hat{S}\mu\right| + \left|\hat{T}\nu - \hat{Q}\nu\right|}\right]\}$$

$$= \max\left\{\mu, \frac{7\mu}{8}, 0, \frac{7\mu}{16}, \frac{7\mu}{8}\left(\frac{8+9\mu}{8+7\mu}\right)\right\}.$$

$$= \mu.$$

And

$$\Delta_2(\mu, v) = 0.$$

Hence

$$\psi(\Delta_1(\mu, v)) - \phi(\Delta_2(\mu, v)) = \psi(\mu) - 0 = \frac{\mu}{2} > \frac{\mu}{8} = \psi\left(d^+(\hat{P}\mu, \hat{Q}\nu)\right).$$

Hence inequality (3.3) satisfied

Case 4. If $\mu \neq 0, \nu \neq 0$.

$$\psi\left(d^+(\hat{P}\mu, \hat{Q}\nu)\right) = \psi(\left|\hat{P}\mu - \hat{Q}\nu\right|) = \psi\left(\left|\frac{\mu}{8} - \frac{\nu}{4}\right|\right) = \left|\frac{\mu}{16} - \frac{\nu}{8}\right|.$$

And

$$\Delta_1(\mu, v) = max\{d^+(\hat{S}\mu, \hat{T}v), d^+(\hat{S}\mu, \hat{P}\mu), d^+(\hat{Q}v, \hat{T}v), \frac{1}{2}[d^+(\hat{S}\mu, \hat{P}\mu) + d^+(\hat{T}v, \hat{Q}v)],$$

$$d^+(\hat{P}\mu, \hat{S}\mu)\left[\frac{1+d^+(\hat{S}\mu, \hat{T}v) + d^+(\hat{Q}v, \hat{P}\mu)}{1+d^+(\hat{P}\mu, \hat{S}\mu) + d^+(\hat{T}v, \hat{Q}v)}\right]\right\}$$

$$= \max\{\left|\hat{S}\mu - \hat{T}\nu\right|, \left|\hat{S}\mu - \hat{P}\mu\right|, \left|\hat{Q}\nu - \hat{T}\nu\right|, \frac{1}{2}\left[\left|\hat{S}\mu - \hat{P}\mu\right| + \left|\hat{T}\nu - \hat{Q}\nu\right|\right],$$

$$\left|\hat{P}\mu - \hat{S}\mu\right| \left[\frac{1+\left|\hat{S}\mu - \hat{T}\nu\right| + \left|\hat{Q}\nu - \hat{P}\mu\right|}{1+\left|\hat{P}\mu - \hat{S}\mu\right| + \left|\hat{T}\nu - \hat{Q}\nu\right|}\right]\}$$

$$= \max\left\{\left|\mu - \frac{\nu}{2}\right|, \frac{7\mu}{8}, \frac{\nu}{4}, \frac{7\mu}{16} + \frac{\nu}{8}, \frac{7\mu}{4}\left|\frac{4+2\mu-\nu}{8-7\mu+2\nu}\right|\right\}.$$

One can easily check that

$$\psi\left(d^+(\hat{P}\mu, \hat{Q}v)\right) \leq \psi(\Delta_1(\mu, \nu)) - \phi(\Delta_2(\mu, \nu)),$$

for all μ, v in $M = [0, 1]$ with $\mu \neq v$.

Hence inequality (3.3) holds for all the cases. Hence all the conditions of Theorem 3.1 hold. Therefore, $\hat{P}, \hat{Q}, \hat{S}$ and $\hat{T}$ have unique common fixed point at $\mu = 0$ in M.

CONCLUSION

In this paper, some common fixed point theorems for four weakly compatible self-maps satisfying the generalized (ϕ,ψ) – weak contraction condition in a complete metric space are proved. Some corollaries are also given by reducing the four maps into two maps.

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