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On a Type of Hilbert C^*-module-valued Convolution Kernel

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Abstract

This paper addresses the study of kernel operators within the framework of Hilbert C^*-modules. Specifically, we investigate kernel operators associated with the convolution kernel type. We study their boundedness, compactness and adjointness.

Keywords: Hilbert C^*-module, kernel operator, convolution kernel.

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INTRODUCTION

A convolution kernel is a function that serves as the kernel for a linear translation-invariant operator. It is a fundamental object as it corresponds to a Fourier multiplier and can also be used to construct Poisson semi-groups [8]. In image processing and computational mathematics, they are implemented as compact matrices applied to digital images to perform operations ranging from basic filtering to sophisticated feature extraction [3] .

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On the other hand, Hilbert C^*-modules, introduced by Kaplansky in the 1950s, are a natural generalization of Hilbert spaces. They are endowed with an inner product that takes values in C^*-algebra instead of the complex field [6]. Many foundational results of Hilbert spaces do not hold for Hilbert modules. For example, the orthogonal decomposition into closed subspaces, a cornerstone of Hilbert space theory, fails in the context of Hilbert C^*-modules [7]. The theory of Hilbert C*-modules is a flourishing field of research. A current trend involves reinvestigating, for Hilbert modules, classical results that are well-established for Hilbert spaces. It is within this framework that we introduce the concept of a convolution kernel taking values in a Hilbert module defined over a locally compact group. We specifically study the operator associated with this type of kernel. More details on Hilbert modules can be found in [6, 7, 2].

The rest of the paper is organized as follows. Section 1 is devoted to notations, concepts and facts which we will need in the sequel. Section 2 contains the main results.

1. PRELIMINARIES

Let $\mathcal{A}$ be a C^*-algebra. Its involution map and its norm are denoted respectively $*$ and $\|\cdot\|_{\mathcal{A}}$.

Definition 1.1. *A pre-Hilbert $\mathcal{A}$-module (or a pre-Hilbert C^*-module over $\mathcal{A}$) is a complex linear space $\mathcal{M}$ equipped with a compatible left $\mathcal{A}$-module structure together with a map $\langle\cdot,\cdot\rangle : \mathcal{M}\times\mathcal{M}\longrightarrow\mathcal{A}$ such that*

1. $\forall x, y, z \in \mathcal{M}$, $\forall \alpha, \beta \in \mathbb{C}$, $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$,
2. $\forall x, y \in \mathcal{M}$, $\forall a \in \mathcal{A}$, $\langle x, ay\rangle = a\langle x, y\rangle$,
3. $\forall x, y \in \mathcal{M}$, $\langle x, y\rangle = \langle y, x\rangle^*$,
4. $\forall x \in \mathcal{M}$, $\langle x, x\rangle \geqslant 0$ *and* $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

In a pre-Hilbert module $\mathcal{M}$, one may define a norm by

$$\|x\|_{\mathcal{M}} = \|\langle x, x\rangle\|_{\mathcal{A}}^{\frac{1}{2}},\ x \in \mathcal{M}. \tag{1}$$

If $\mathcal{M}$ is complete with respect to the above norm, then $\mathcal{M}$ is called a Hilbert $\mathcal{A}$-module (or a Hilbert C^*-module over $\mathcal{A}$).

Let us provide some examples.

Examples 1.2. 1. Let $\mathcal{A}$ be a C^*-algebra.

The space $\ell_2(\mathcal{A}) = \{(x_n)_{n\in\mathbb{N}} \subset \mathcal{A}; \sum_{n=1}^{+\infty} x_n^* x_n$ converges in $\mathcal{A}\}$ is a Hilbert $\mathcal{A}$-module with respect to the $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle$ defined by $\langle (x_n),(y_n)\rangle = \sum_{n=0}^{+\infty} x_n^* y_n$.

2. Let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. Let us denote by $L^2(G,\mathcal{M})$ the set of $\mathcal{M}$-valued Bochner square integrable functions on G. It was shown in [1] that $L^2(G,\mathcal{M})$ is a Hilbert $\mathcal{A}$-module with respect to the $\mathcal{A}$-valued inner product

$$\langle f,g\rangle_{L^2(G,\mathcal{M})} = \int_G \langle f(x),g(x)\rangle d\lambda(x). \tag{2}$$

We recall the following theorem that we may use.

Theorem 1.3. *([6, pages 3, 4])*

1. *For all $x,y\in\mathcal{M}$, $\langle y,x\rangle\langle x,y\rangle \leqslant \|\langle x,x\rangle\|\langle y,y\rangle$.*
2. *For all $x,y\in\mathcal{M}$, $\|\langle x,y\rangle\|_{\mathcal{A}} \leqslant \|x\|_{\mathcal{M}}\|y\|_{\mathcal{M}}$.*
3. *For all $x\in\mathcal{M}$, $a\in\mathcal{A}$, $\|ax\|_{\mathcal{M}} \leqslant \|a\|_{\mathcal{A}}\|x\|_{\mathcal{M}}$.*

2. MAIN RESULTS

Let G be a locally compact Hausdorff group with a fixed left Haar measure. We assume that whenever G is a compact group, then its Haar measure is normalized in such a way that $\lambda(G)=1$. Throughout this paper, for any Banach space $(X,\|\cdot\|_X)$, we denote by $L^p(G,X)$, $1\leqslant p<\infty$, the space of X-valued Bochner p-th power integrable (class of) functions on G with respect to the Haar measure λ. The set $L^p(G,X)$ is a Banach space with respect to the norm

$$\|f\|_{L^p(G,X)} = \left(\int_G \|f(x)\|_X^p d\lambda(x)\right)^{\frac{1}{p}},\ f\in L^p(G,X). \tag{3}$$

Definition 2.1. *Let G be a locally compact group and let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. We say that the linear operator $T: L^1(G,\mathcal{M}) \longrightarrow L^1(G,\mathcal{A})$ is a kernel operator if there exists a $\lambda\times\lambda$-measurable function $k_T: G\times G\to\mathcal{M}$ such that*

$$Tf(x) = \int_G \langle k_T(x,y), f(y)\rangle d\lambda(y),\ x\in G. \tag{4}$$

and

$$\int_G \int_G \|\langle k_T(x,y), f(y)\rangle\|_{\mathcal{A}} d\lambda(x) d\lambda(y) < \infty. \tag{5}$$

The function k_T is called the kernel of the operator T. Let us denote by $\mathcal{K}(G,\mathcal{M})$ the set of kernel operators $T : L^1(G,\mathcal{M}) \to L^1(G,\mathcal{A})$.

In this framework, in contrast to [4], the domain of the kernel operator T is a subspace of $L^1(G,\mathcal{M})$ and is defined by

$$\begin{aligned} dom(T) \;&= \Big\{g \in L^1(G,\mathcal{M}) :\; (x,y) \mapsto \langle k_T(x,y), g(y)\rangle \in L^1(G\times G,\mathcal{A}) \text{ and if} \\ &\forall x \in G,\, Tg(x) = \int_G \langle k_T(x,y), g(y)\rangle d\lambda(y) \text{ then } Tg \in L^1(G,\mathcal{A})\Big\}. \end{aligned} \tag{6}$$

The range of T is denoted by

$$ran(T) = \{Tg :\; g \in dom(T)\}. \tag{7}$$

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces, and let $T : E \to F$ be a linear operator. The operator norm of T is denoted by

$$\|T\| = \sup\{\|Tf\|_F :\; \|f\|_E \leqslant 1\}. \tag{8}$$

2.1. Convolution kernel

In the following, let us assume that G is a compact group and its Haar measure λ is a normalized invariant measure. Let us introduce the kernel operator for a convolution kernel in the framework of Hilbert C^*-modules. Let $\phi : G \to \mathcal{M}$ be a measurable continuous function. Put T_ϕ as the kernel operator associated with the kernel $k_\phi : G\times G \to \mathcal{M}$ defined by $k_\phi(x,y) = \phi(xy^{-1})$ for any $(x,y) \in G\times G$. We will call k_ϕ a convolution kernel. The following results are inspired from [4].

Theorem 2.2. *(i) The map T_ϕ is well-defined, that is, the map $(x,y) \mapsto \langle k_\phi(x,y), f(y)\rangle$ belongs to $L^1(G\times G,\mathcal{A})$ for any $f \in L^1(G,\mathcal{M})$ and $T_\phi f \in L^1(G,\mathcal{A})$.*

(ii) $T_\phi : L^1(G,\mathcal{M}) \to L^1(G,\mathcal{A})$ is bounded with

$$||T_\phi|| \leqslant ||\phi||_{L^1(G,\mathcal{M})}. \tag{9}$$

Proof. (i) Let $f \in L^1(G, \mathcal{M})$. We have

$$\begin{aligned}
\int_G \int_G ||\langle k_\phi(x,y), f(y)\rangle||_{\mathcal{A}} d\lambda(x) d\lambda(y) &\leqslant \int_G \int_G ||k_\phi(x,y)||_{\mathcal{M}} ||f(y)||_{\mathcal{M}} d\lambda(x) d\lambda(y) \\
&= \int_G ||f(y)||_{\mathcal{M}} \left(\int_G ||\phi(xy^{-1})||_{\mathcal{M}} d\lambda(x) \right) d\lambda(y) \\
&= \int_G ||f(y)||_{\mathcal{M}} ||\phi||_{L^1(G,\mathcal{M})} d\lambda(y) \\
&= ||\phi||_{L^1(G,\mathcal{M})} ||f||_{L^1(G,\mathcal{M})} < \infty.
\end{aligned}$$

Hence, the map $(x,y) \mapsto \langle k_\phi(x,y), f(y)\rangle$ belongs to $L^1(G \times G, \mathcal{A})$.

(ii) Let $f \in L^1(G, \mathcal{M})$. We have

$$\begin{aligned}
\int_G \|T_\phi f(x)\|_{\mathcal{A}} d\lambda(x) &= \int_G \left\| \int_G \langle \phi(xy^{-1}), f(y)\rangle d\lambda(y) \right\|_{\mathcal{A}} d\lambda(x) \\
&\leqslant \int_G \int_G \|\phi(xy^{-1})\|_{\mathcal{M}} \|f(y)\|_{\mathcal{M}} d\lambda(y) d\lambda(x) \\
&\leqslant \|\phi\|_{L^1(G,\mathcal{M})} \|f\|_{L^1(G,\mathcal{M})} < \infty.
\end{aligned}$$

Hence, $T_\phi f \in L^1(G, \mathcal{A})$ and $||T_\phi f||_{L^1(G,\mathcal{A})} \leqslant ||\phi||_{L^1(G,\mathcal{M})} ||f||_{L^1(G,\mathcal{M})}$. Therefore, $T_\phi : L^1(G, \mathcal{M} \to L^1(G, \mathcal{A})$ is bounded and $||T_\phi|| \leqslant ||\phi||_{L^1(G,\mathcal{M})}$. □

Theorem 2.3. *Let G be a compact group and let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. Then, the kernel $k_\phi \in L^2(G \times G, \mathcal{M})$ and $\|k_\phi\|_{L^2(G\times G,\mathcal{M})} = \|\phi\|_{L^2(G,\mathcal{M})}$.*

Proof. We have

$$\begin{aligned}
\int_G \int_G ||k_\phi(x,y)||^2_{\mathcal{M}} d\lambda(x) d\lambda(y) &= \int_G \int_G ||\phi(xy^{-1})||^2_{\mathcal{M}} d\lambda(x) d\lambda(y) \\
&= \int_G \int_G ||\phi(y)||^2_{\mathcal{M}} d\lambda(y) d\lambda(x) \\
&= \int_G ||\phi||^2_{L^2(G,\mathcal{M})} d\lambda(x) \\
&= ||\phi||^2_{L^2(G,\mathcal{M})} < \infty.
\end{aligned}$$

Hence, $k_\phi \in L^2(G \times G, \mathcal{M})$ and $\|k_\phi\|_{L^2(G\times G,\mathcal{M})} = \|\phi\|_{L^2(G,\mathcal{M})}$. □

Theorem 2.4. *Let G be a compact group and let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. Then, $T_\phi : L^1(G, \mathcal{M}) \to L^1(G, \mathcal{A})$ is a compact operator.*

Proof. Given that G is a compact group, $L^2(G, \mathcal{M})$ is dense in $L^1(G, \mathcal{M})$. Therefore, there exists a sequence $(\phi_n)_n$ of $L^2(G, \mathcal{M})$ such that $||\phi - \phi_n||_{L^1(G,\mathcal{M})}$ converges to

0 as n goes to $+\infty$. From inequality (9), we see that $||T_\phi - T_{\phi_n}|| = ||T_{\phi-\phi_n}|| \leqslant ||\phi - \phi_n||_{L^1(G,\mathcal{M})}$ converges to 0 as n goes to $+\infty$. Since for all $n \in \mathbb{N}$, $T_{\phi_n} : L^2(G,\mathcal{M}) \to L^1(G,\mathcal{A})$ is compact, it comes that T_ϕ is compact. $\square$

Theorem 2.5. *Let G be a compact group and let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. Then, the map $G \times G \to \mathbb{R}_+$, $(x, y) \mapsto ||k_\phi(x,y)||_{\mathcal{M}}||f(y)||^2_{\mathcal{M}}$ is integrable and*

$$\int_G\int_G ||k_\phi(x,y)||_{\mathcal{M}}||f(y)||^2_{\mathcal{M}}d\lambda(x)d\lambda(y) = ||\phi||_{L^1(G,\mathcal{M})}||f||^2_{L^2(G,\mathcal{M})}. \tag{10}$$

Proof. Given $f \in L^2(G,\mathcal{M})$, we have

$$\begin{aligned}\int_G\int_G ||k_\phi(x,y)||_{\mathcal{M}}||f(y)||^2_{\mathcal{M}}d\lambda(x)d\lambda(y) &= \int_G ||f(y)||^2_{\mathcal{M}}\left(\int_G ||\phi(xy^{-1})||_{\mathcal{M}}d\lambda(x)\right)d\lambda(y)\\ &= ||\phi||_{L^1(G,\mathcal{M})}\int_G ||f(y)||^2_{\mathcal{M}}d\lambda(y)\\ &= ||\phi||_{L^1(G,\mathcal{M})}||f||^2_{L^2(G,\mathcal{M})}.\end{aligned}$$

$\square$

Theorem 2.6. *Let G be a compact group and let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. If $f \in dom(T) \subset L^2(G,\mathcal{M})$, then $T_\phi f \in L^2(G,\mathcal{A})$. Moreover, the kernel operator $T_\phi : dom(T) \subset L^2(G,\mathcal{M}) \to L^2(G,\mathcal{A})$ is bounded and*

$$||T_\phi|| \leqslant ||\phi||_{L^1(G,\mathcal{M})}. \tag{11}$$

Proof. Let $f \in dom(T) \subset L^2(G,\mathcal{M})$, we have

$$\int_G ||T_\phi f(x)||^2_{\mathcal{A}}d\lambda(x) = \int_G ||\int_G \langle k_\phi(x,y), f(y)\rangle d\lambda(y)||^2_{\mathcal{A}}d\lambda(x).$$

Also,

$$\begin{aligned}\left|\left|\int_G \langle k_\phi(x,y), f(y)\rangle d\lambda(y)\right|\right|_{\mathcal{A}} &\leqslant \int_G ||k_\phi(x,y)||_{\mathcal{M}}||f(y)||_{\mathcal{M}}d\lambda(y)\\ &= \int_G ||k_\phi(x,y)||^{\frac{1}{2}}_{\mathcal{M}}\left(||k_\phi(x,y)||^{\frac{1}{2}}_{\mathcal{M}}||f(y)||_{\mathcal{M}}\right)d\lambda(y)\\ &\leqslant \left(\int_G ||k_\phi(x,y)||_{\mathcal{M}}d\lambda(y)\right)^{\frac{1}{2}}\\ &\quad \left(\int_G ||k_\phi(x,y)||_{\mathcal{M}}||f(y)||^2_{\mathcal{M}}d\lambda(y)\right)^{\frac{1}{2}}\\ &= ||\phi||^{\frac{1}{2}}_{L^1(G,\mathcal{M})}\left(\int_G ||k_\phi(x,y)||_{\mathcal{M}}||f(y)||^2_{\mathcal{M}}d\lambda(y)\right)^{\frac{1}{2}}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\int_G ||T_\phi f(x)||_{\mathcal{A}}^2 d\lambda(x) &\leqslant \int_G || \int_G \langle k_\phi(x,y), f(y)\rangle d\lambda(y)||_{\mathcal{A}}^2 d\lambda(x)\\ &\leqslant \int_G \left(||\phi||_{L^1(G,\mathcal{M})}^{\frac{1}{2}} \left(\int_G ||k_\phi(x,y)||_{\mathcal{M}} ||f(y)||_{\mathcal{M}}^2 d\lambda(y)\right)^{\frac{1}{2}}\right)^2 d\lambda(x)\\ &= ||\phi||_{L^1(G,\mathcal{M})} \int_G\int_G ||k_\phi(x,y)||_{\mathcal{M}} ||f(y)||_{\mathcal{M}}^2 d\lambda(x) d\lambda(y).\end{aligned}$$

From equality (10) of Theorem 2.5, we have

$$\int_G ||T_\phi f(x)||_{\mathcal{A}}^2 d\lambda(x) \leqslant ||\phi||_{L^1(G,\mathcal{M})}^2 ||f||_{L^2(G,\mathcal{M})}^2 < \infty.$$

Hence, $T_\phi f \in L^2(G,\mathcal{A})$. Furthermore $||T_\phi f||_{L^2(G,\mathcal{A})} \leqslant ||\phi||_{L^1(G,\mathcal{M})} ||f||_{L^2(G,\mathcal{M})}$. It follows that $T_\phi : dom(T) \subset L^2(G,\mathcal{M}) \to L^2(G,\mathcal{A})$ is bounded and $||T_\phi|| \leqslant ||\phi||_{L^1(G,\mathcal{M})}$. □

Theorem 2.7. *Le G be a compact group and let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module.Then, the kernel operator $T_\phi : dom(T) \subset L^2(G,\mathcal{M}) \to L^2(G,\mathcal{A})$ is adjointable.*

Proof. Let $f \in L^2(G,\mathcal{M})$ and $g \in L^2(G,\mathcal{A})$. We have

$$\begin{aligned}\langle T_\phi f, g\rangle_{L^2(G,\mathcal{A})} &= \int_G (T_\phi f(x))^* g(x) d\lambda(x)\\ &= \int_G \left(\int_G \langle \phi(xy^{-1}), f(y)\rangle d\lambda(y)\right)^* g(x) d\lambda(x)\\ &= \int_G \left(\int_G \langle \phi(xy^{-1}), f(y)\rangle^* d\lambda(y)\right) g(x) d\lambda(x)\\ &= \int_G \left(\int_G \langle f(y), g(x)\phi(xy^{-1})\rangle d\lambda(y)\right) d\lambda(x)\\ &= \int_G \int_G \langle f(y), g(x)\phi(xy^{-1})\rangle d\lambda(y) d\lambda(x)\\ &= \int_G \langle f(y), \int_G g(x)\phi(xy^{-1}) d\lambda(x)\rangle d\lambda(y).\end{aligned}$$

Put $S_\phi g(y) = \int_G g(x)\phi(xy^{-1}) d\lambda(x)$ for all $y \in G$. With a similar reasoning to the proof of Theorem 2.6, $S_\phi g \in L^2(G,\mathcal{M})$ and the map $S_\phi : L^2(G,\mathcal{A}) \to L^2(G,\mathcal{M})$ is an $\mathcal{A}$-linear bounded operator. Hence,

$$\begin{aligned}\langle T_\phi f, g\rangle_{L^2(G,\mathcal{A})} &= \int_G \langle f(y), S_\phi g(y)\rangle d\lambda(y) \\ &= \langle f, S_\phi g\rangle_{L^2(G,\mathcal{M})}.\end{aligned}$$

It follows that $T_\phi : dom(T) \subset L^2(G,\mathcal{M}) \to L^2(G,\mathcal{A})$ is adjointable and its adjoint is the map S_ϕ. □

REFERENCES

[1] Ali S. T., Bhattacharyya T., Roy S. S., (2011), ”Coherent states on Hilbert modules”, J. Phys. A: Math. Theor., 44, 275202, 16pp.

[2] Frank, M., (2025), ”Multiplier modules of Hilbert C*-modules revisited”, preprint arXiv:2502.17959.

[3] Gonzalez, R. C., Woods, R. E., (2018), ”Digital Image Processing”, (4th ed.). Pearson.

[4] Halmos P. R., Sunder V. S., (1978), ”Bounded integral operators on L^2 spaces”, Springer-Verlag.

[5] Kaplansky I., (1953), ”Modules over operator algebras”, Am. J. Math., 75(4): 839-853.

[6] Lance E. C., (1995), ”Hilbert C^*-Modules, A toolkit for operator algebraists”, Cambridge University Press.

[7] Manuilov V. M., Troitsky E. V., (2005), ”Hilbert C^*-modules”, AMS.

[8] Stein, E. M., (1970), ”Singular Integrals and Differentiability Properties of Functions”, Princeton University Press.