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Positive Partially Ordered $\Gamma-$ Semirings

***Tilak Raj Sharma, **Rajesh Kumar and ***Sanjeev Kumar**

*Department of Mathematics,, *Himachal Pradesh University, Regional Centre Dharamshala,, Himachal Pradesh (India) , **Government Degree College Dharamshala, (HP) (India) , ***Government (Girls) Senior Secondary School Indora, (HP) (India), Gmail: trpangotra@gmail.com*

Abstract

The main objective of this paper is to derive some of the results of partially ordered $\Gamma-$ semiring by using the conditions of positive, additive idempotent, centreless, simple, commutative, strong identity on partially ordered $\Gamma-$ semirings.

Keywords: $\Gamma-$ semiring, partially ordered $\Gamma-$semirings, commutative $\Gamma-$ semiring, additive idempotent.

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1. INTRODUCTION

An algebraic structure along with two binary operations addition and multiplication, where addition is commutative and multiplication is distributive over addition is called semiring. Semirings not only have significant applications in different fields such as optimization theory, generalized fuzzy computation, and automata theory in computer science, but are fairly interesting generalizations of rings and bounded distributive lattices.

Semirings provide the most natural common generalization of the theory of rings, also they provide an important tool in the development of different branches of science. But the set of all non-positive integers and the set of all $m \times n$ matrices do not form semirings because multiplication is not a binary operation for the above sets. So another mathematical operation came into existence, which is known as $\Gamma-$ Semiring. The notion of Γ in algebra was introduced by N. Nobusawa [6] in 1964. In 1981, M. K. Sen [7] introduced the notion of $\Gamma-$ semigroup. In 1995, the concept of $\Gamma-$semiring was introduced by Rao [5] as a generalization of $\Gamma-$semiring, ternary semiring, semigroups, and semirings. After this T. K. Dutta and S. K. Sardar[1], H. Hedayati and K. P. Shum[3], Sharma and Gupta[9], Sharma and Ranote [11] and many others obtained interesting results on $\Gamma-$ semirings.

In this paper, we here consider some conditions of positive, additive idempotent, centreless, simple, commutative, strong identity, etc. on positive partially ordered $\Gamma-$ semirings and generalize some of the results of semiring [2] to various $\Gamma-$ semirings like Gel'fand $\Gamma-$ semiring, complimented $\Gamma-$ semiring, multiplicatively cancellative commutative $\Gamma-$ semiring.

2. PRELIMINARIES AND EXAMPLES

Recall that if (R, +) and (Γ, +) be two commutative semigroups then R is called a $\Gamma-$ semiring if there exists a mapping R $\times$ Γ $\times$ R $\rightarrow$ R denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying(i) $x\alpha(y+z) = x\alpha y + x\alpha z$. (ii) $(y+z)\alpha x = y\alpha x + z\alpha x$. (iii) $x(\alpha+\beta)z = x\alpha z + x\beta z$. (iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Let A and B be semirings and let $R = Hom(A, B)$ and $\Gamma = Hom(B, A)$ denote the sets of homomorphisms from A to B and B to A respectively. Then R is a $\Gamma-$ semiring with operations of pointwise addition and composition of mappings. Further, let M be a $\Gamma-$ ring and let R be the set of ideals of M. Define addition in the natural way and if $A, B \in R$, $\gamma \in \Gamma$, let $A\gamma B$ denote the ideal generated by $\{x\gamma y | x, y \in M\}$. Then R is a $\Gamma-$ semiring. A $\Gamma-$ semiring R is said to be commutative if $x\gamma y = y\gamma x$ for all $x, y \in R$ and for all $\gamma \in \Gamma$. A $\Gamma-$ semiring R is said to have a zero element if $0\gamma x = 0 = x\gamma 0$ and $x + 0 = x = 0 + x$ for all $x \in R$ and $\gamma \in \Gamma$. R is said to have an identity element if there exists $\gamma \in \Gamma$ such that $1\gamma x = x = x\gamma 1$ for all $x \in R$. R is said to have a strong identity element if for all $x \in R$, $1\alpha x = x = x\alpha 1$ for all $\alpha \in \Gamma$. A non empty subset S of a $\Gamma-$ semiring R is said to be a sub $\Gamma-$ semiring of R if $(S, +)$ is a sub semigroup of $(R, +)$ and $x\gamma y \in S$ for all $x, y \in S$ and $\gamma \in \Gamma$. An element x of a $\Gamma-$ semiring R is said to be additive idempotent if $x = x + x$. If every element of R is additive idempotent then R is called additive idempotent $\Gamma-$ semiring. It is denoted by $I^+(\Gamma R)$. A $\Gamma-$ semiring R is simple if and only if $x + 1 = 1 = 1 + x$ for all $x \in R$. Simple $\Gamma-$ semiring are additive idempotent but the converse is not true.

Throughout this paper, R will denote a $\Gamma-$ semiring with zero elements "0" and identity element "1" unless otherwise stated.

3. POSITIVE PARTIALLY ORDERED $\Gamma-$ SEMIRINGS

Definition 3.1. *Let R be a $\Gamma-$ semiring. Then R is called partially ordered $\Gamma-$ semiring if and only if there exists a partial order relation $' \leq'$ on R satisfying the following conditions if $x \leq y$ and $z \geq 0$ then*

(i) $x + z \leq y + z$

(ii) $x\alpha z \leq y\alpha z$

(iii) $z\alpha x \leq z\alpha y,$ *for all* $x, y, z \in R$ *and* $\alpha \in \Gamma$.

If the relation $\leq$ is a total order then R is totally ordered $\Gamma-$ semiring.

Example 3.2. *[4] Let $R = [0, 1], \Gamma = \mathbb{N}$. Define $x + y = max\{x, y\}$ and $x\alpha y = min\{x, \alpha, y\}$ for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is a partially ordered $\Gamma-$ semiring with respect to usual ordering.*

Example 3.3. *let R be a $\Gamma-$ semiring and let $\{\leq_i |i \in \Omega\}$ be a family of partial order relations on R each of which turns R into a partially ordered $\Gamma-$ semirings. Then R is a partially ordered $\Gamma-$ semiring concerning the relation $' \leq ''$ defined by $x \leq y$ if and only if $x \leq_i y$ for all $i \in \Omega$.*

Definition 3.4. *A partially ordered $\Gamma-$ semiring in which every two elements are comparable is said to be totally ordered $\Gamma-$ semiring.*

Definition 3.5. *An element x of a $\Gamma-$ semiring R is a unit if and only if there exists an element y of R and $\alpha \in \Gamma$ satisfying $x\alpha y = 1 = y\alpha x$. The element y of R is called the inverse of x in R.*

Definition 3.6. *An element x of a $\Gamma-$ semiring R is a strong unit if and only if there exists an element y of R satisfying $x\alpha y = 1 = y\alpha x$ for all $\alpha \in \Gamma$. The element y of R is called the strong inverse of x in R.*

Let us denote the set of all elements of R having units or strong units by $U(\Gamma R)$. This set is non-empty since $1 \in U(\Gamma R)$ and is not all of R.

Definition 3.7. *A $\Gamma-$ semiring R is a division $\Gamma-$ semiring if and only if every non-zero element of R is a unit.*

Definition 3.8. *Commutative division $\Gamma-$ semirings are $\Gamma-$ semifields.*

Definition 3.9. *[12] Let R be a $\Gamma-$ semiring and define $G_F(R) = \{r \in R|\ 1 + r \in U(\Gamma R)\}$. Then $G_F(R)$ is a Gel'fand $\Gamma-$ semiring if and only if $R = G_F(R)$.*

Definition 3.10. *An element x of a partially ordered $\Gamma-$ semiring R is transitive if and only if $x\alpha x \leq x, \alpha \in \Gamma$.*

Definition 3.11. *An element x of a partially ordered $\Gamma-$ semiring R is positive if and only if $x \geq 0$. This condition is equivalent to the condition that $x + r \geq r$ for all $r \in R$.*

Definition 3.12. *A $\Gamma-$ semiring R with zero elements is said to satisfy cancellation law if for all $a, b, c \in R$ and $\alpha \in \Gamma$ we have that $a \neq 0$, $a\alpha b = a\alpha c$ and $b\alpha a = c\alpha a$ implies $b = c$.*

Recall the following definitions from [10].

Definition 3.13. *[10] Let x, y be elements of a $\Gamma-$ semiring R then x is $\Gamma-$ interior y denoted by $x\nabla y$ if and only if there exist an element $z \in R, \alpha \in \Gamma$ such that $x\alpha z = z\alpha x = 0$, and $z + y = 1$.*

Definition 3.14. *[10] An element x of a $\Gamma-$ semiring R is complemented if and only if $x\nabla x$. That is there exists an element $y \in R, \alpha \in \Gamma$ such that $x\alpha y = y\alpha x = 0$ and $x + y = 1$, this element $y \in R$ is the complement of R.We will denote the complement of x by $x^{\perp}$.*

Theorem 3.15. *Let x be an element of a partially ordered $\Gamma-$ semiring satisfying $x \leq y$ for all $y \in R$ then $x \in I^{+}(\Gamma R)$.*

Proof. Since $x \leq y$ for all $y \in R$, so we have $x \leq 0$ and therefore $x \leq x + x \leq x + 0 = x$. Hence, $x + x = x$. □

Theorem 3.16. *Let R be a positive partially ordered $\Gamma-$ semiring. Then R is centreless.*

Proof. Let R be a positive partially ordered $\Gamma-$ semiring. If $x, y \in R$ then $x \geq 0, y \geq 0$. Now, $x + y \geq x + 0 = x \geq 0$. Thus, $x + y = 0$ implies that $0 \geq x \geq 0$. This implies that $x = 0$. Similarly, $y = 0$. □

The following result is proved in [9].

Theorem 3.17. *Let R be a $\Gamma-$semiring. Then*

(i) R is simple if and only if $x = x + x\alpha y$, for all $x, y \in R, \alpha \in \Gamma$

(ii) R is simple if and only if $x = x + y\alpha x$, for all $x, y \in R, \alpha \in \Gamma$

(iii) R is simple if and only if $x\alpha y = x\alpha y + (x\beta z)\alpha y$, for all $x, y, z \in R, \alpha, \beta \in \Gamma$.

Theorem 3.18. *Let* R *be a positive partially ordered Gel'fand* $\Gamma-$ *semiring then for each* $x, y \in R$ *there exist units* $u, v \in U(\Gamma R)$ *such that* $x\alpha y \leq x\alpha u$ *and* $x\alpha y \leq x\alpha v, \alpha \in \Gamma$*. Further, if* R *is a strong simple* $\Gamma-$ *semiring then* $x\alpha y \leq y$ *and* $y\alpha x \leq x$ *for all* $x, y \in R, \alpha \in \Gamma$

Proof. Let $x, y \in R$ then $u = 1 + x$ and $v = 1 + y$ are units of R, as R is a Gel'fand $\Gamma-$ semiring. Now, since R is positive, so we have $x \leq x + 1$ and $y \leq y + 1$. This implies that $x\alpha y \leq (x+1)\alpha y = u\alpha y$ and $x\alpha y \leq x\alpha(y+1) = x\alpha v$ Moreover, if R is a strong simple $\Gamma-$ semiring then by Theorem 3.17, $x\alpha y \leq (x+1)\alpha y = 1\alpha y = y$ and $y\alpha x \leq (y+1)\alpha x = 1\alpha x = x$ for all $x, y \in R, \alpha \in \Gamma$. □

Theorem 3.19. *Let* R *be a positive, simple and commutative* $\Gamma-$ *semiring and* $x_1, x_2, x_3, ..., x_n$ *are elements of* R*. If* $1 \leq h \leq k \leq n$ *are indices such that* $x_h\alpha_h x_k = 0$ *then* $x_1\alpha_1 x_2\alpha_2 x_3\alpha_3 ...\alpha_n x_n = 0$.

Proof. Let $y = x_1\alpha_1 x_2\alpha_2 x_3\alpha_3 ...\alpha_{h-1} x_h$ and

$$z = x_{h+1}\alpha_{h+1} x_{h+2}\alpha_{h+2} x_{h+3}\alpha_{h+3} ...\alpha_{k-1} x_k.$$

Then by Theorem 3.17, $0 \leq y \leq x_h$ and $0 \leq z \leq x_k$.
This implies that $0 \leq y\alpha_h z \leq x_h\alpha_h x_k = 0$. Therefore, $y\alpha_h z = 0$.
Now,

$$\begin{aligned}
&x_1\alpha_1 x_2\alpha_2 x_3\alpha_3 ...x_h\alpha_h x_{h+1} ...\alpha_{n-1} x_n \\
&\quad = (x_1\alpha_1 x_2\alpha_2 x_3 ...\alpha_{h-1} x_h)\alpha_h (x_{h+1}\alpha_{h+1} x_{h+2}\alpha_{h+2} ...\alpha_{k-1} x_k)\alpha_k x_{k+1} ...\alpha_{n-1} x_n \\
&\quad = x_1\alpha_1 x_2\alpha_2 x_3 ...\alpha_{h-1}(x_h\alpha_h x_k)\alpha_{h+1} x_{h+1}\alpha_{h+2} ...\alpha_{k-1} x_{k-1}\alpha_k x_{k+1} ...\alpha_{n-1} x_n \\
&\quad = 0
\end{aligned}$$

□

Theorem 3.20. *Let* R *be an additively idempotent* $\Gamma-$ *semiring with strong identity. Then* R *is partially ordered by the relation* $x \leq y$ *if and only if* $x + y = y$*. Moreover,* R *is positive and a join semilattice with* $x \vee y = x + y$*. Further, if* $x, y \in U(\Gamma R)$ *then* $x \geq y$ *if and only if* $w \leq t$*, where* w *and* t *are inverses of* x *and* y *respectively.*

Proof. Let R be an additively idempotent $\Gamma-$ semiring with strong identity. Now, $x \leq y$ if and only if $x+y = y$ if and only if $x+y+z = y+z$ if and only if $x+y+z+z = y+z$ if and only if $x+z+y+z = y+z$ if and only if $x+z \leq y+z$. Again, let $x \leq y$ and $z \geq 0$ if and only if $x + y = y$ if and only if $(x+y)\alpha z = y\alpha z$ if and only if $x\alpha z + y\alpha z = y\alpha z$ if and only if $x\alpha z \leq y\alpha z$, $\alpha \in \Gamma$. Thus, R is partially ordered $\Gamma-$ semiring. Clearly $x \geq 0$ for all $x \in R$. So, R is positive.
Moreover, R is additively idempotent, so $x+y = x+y$ if and only if $x+(x+y) = x+y$ if and only if $x \leq x+y$. Similarly, $y \leq x+y$ for all $x, y \in R$. Let $z \in R$ be such that $x \leq z$ and $y \leq z$. Then $x+z = z$ and $y+z = z$. Therefore, $(x+y)+z = x+(y+z) =$

$x+z=z$. This implies that $x+y\le z$. Thus, $x\le z, y\le z$ implies that $x+y\le z$. So $x+y=x\vee y$. Finally, if $x,y\in U(\Gamma R)$ then $y\le x$ if and only if $y+x=x$. Now, $t=t\beta(x\alpha w)=t\beta(y+x)\alpha w=(t\beta y)\alpha w+t\beta(x\alpha w)=1\alpha w+t\beta 1=w+t$, where w and t are inverses of x and y respectively such that $x\alpha w=w\alpha x=1$ and $y\alpha t=t\alpha y=1$. Hence, $w\le t$. $\square$

Corollary 3.21. *Let R be a simple $\Gamma-$ semiring with strong identity. Then every $\Gamma-$ semiring is partially ordered by the relation $x\le y$ if and only if $x+y=y$ and satisfies $0\le x\le 1$ for all $x\in R$.*

Theorem 3.22. *Let R be an additively idempotent $\Gamma-$ semiring with strong identity. Then $S=\{x\in R\mid 0\le x\le 1\}$ is a sub $\Gamma-$ semiring of R.*

Proof. Clearly $0,1\in R$. If $x,y\in S,\alpha\in\Gamma$ then $0\le x+y\le 1+1=1$ and $0=x\alpha 0\le x\alpha y\le x\alpha 1=x\le 1$. Thus, $x+y, x\alpha y\in S$. $\square$

From now onwards, whenever we consider additively idempotent $\Gamma-$ semirings with a strong identity, we will assume that they are partially ordered by the relation given in Theorem 3.20. In particular, this will be also true for simple $\Gamma-$ semirings.

The following results are proved in [11].

Theorem 3.23. *[11] Let R be a $\Gamma-$ semiring. If R is multiplicatively cancellative, additively idempotent commutative $\Gamma-$ semiring. Then*
$((x+y)\alpha)^{n-1}(x+y)=(x\alpha)^{n-1}x+(y\alpha)^{n-1}y$ for all $x,y\in R,\ \alpha\in\Gamma$ and all positive integers n.

Corollary 3.24. *[11] If R is an additively idempotent multiplicatively cancellative commutative $\Gamma-$ semiring and if $x_1,x_2,.....x_t\in R,\ \alpha\in\Gamma$ then*
$((\sum_{i=1}^{t}x_i)\alpha)^{n-1}(\sum_{i=1}^{t}x_i)=\sum_{i=1}^{t}(x_i\alpha)^{n-1}x_i$, for all positive integers n.

As an immediate of Theorem 3.23 and Corollary 3.24, we have

Theorem 3.25. *Let R be a multiplicatively cancellative additively idempotent commutative $\Gamma-$ semiring. If $x_1,x_2,.....x_n\in R$ then $x_1\alpha x_2\alpha...\alpha x_n\le\sum_{i=1}^{n}(x_i\alpha)^{n-1}x_i$.*

Theorem 3.26. *Let R be an additively idempotent and simple $\Gamma-$ semiring. Then $x\le(y\nabla z)\le t$ in R implies that $x\nabla t$ for any $x,y,z,t\in R$.*

Proof. Since $y\nabla z$, so there exists an element $w \in R, \alpha \in \Gamma$ such that $y\alpha w = w\alpha y = 0$ and $w + z = 1$. Again, by Theorem 3.20, $x \leq y$ if and only if $x + y = y$. By Theorem 3.17, we have $y = x\beta y + y$ for all $x, y \in R, \beta \in \Gamma$. Thus, $x + y = x\beta y + y$. Now, $x\alpha w = x\alpha w + 0 = x\alpha w + y\alpha w = (x + y)\alpha w = y\alpha w = (x\beta y + y)\alpha w = (x\beta y)\alpha w + y\alpha w = 0$. Similarly, $w\alpha x = 0$. Again, $z \leq t$ if and only if $z + t = t$. Thus, $w + t = w + z + t = 1 + t = 1$, since R is simple. Hence $x\nabla t$. □

Theorem 3.27. *Let R be a positive, commutative and simple $\Gamma-$ semiring. Then $x\nabla y$ and $z\nabla t$ in R implies that $(x\alpha z + z\alpha x)\nabla(y\beta t)$ for any $x, y, z, t \in R$ and $\alpha, \beta \in \Gamma$.*

Proof. Since, $x\nabla y$ and $z\nabla t$, so there exist elements r, s of R and $\alpha, \beta \in \Gamma$ such that $x\alpha r = r\alpha x = 0 = z\beta s = s\beta z$ and $r + y = 1 = s + t$, for $x, y, z \in R$. Let $w = y\delta s + r$, for some $w \in R$, $\delta \in \Gamma$. Now, $w\beta(x\alpha z + z\alpha x) = (y\delta s + r)\beta(x\alpha z + z\alpha x) = (y\delta s)\beta(x\alpha z) + (y\delta s)\beta(z\alpha x) + r\beta(x\alpha z) + r\beta(z\alpha x) = y\delta(s\beta z)\alpha x + y\delta(s\beta z)\alpha x + r\beta(z\alpha x) + z\beta(r\alpha x) = 0$. Similarly,$(x\alpha z + z\alpha x)\beta w = 0$. Again, $w + y\delta t = y\delta(t + s) + r = y\delta 1 + r = y + r = 1$. Hence, $(x\alpha z + z\alpha x)\nabla y\delta t$. □

Theorem 3.28. *Let R be a positive, commutative, and simple $\Gamma-$ semiring with strong identity. Then $x, z\nabla y$ implies that $(x + z)\nabla y$ for any $x, y, z \in R$.*

Proof. Since $x\nabla y$ and $z\nabla y$, so there exist elements $r, s \in R, \alpha \in \Gamma$ such that $x\alpha r = r\alpha x = z\alpha s = s\alpha z = 0$ and $r + y = s + y = 1$. let $t = r\delta s, \delta \in \Gamma$. Now, R is positive, therefore we have $0 \leq t\alpha(x + z) = (r\delta s)\alpha(x + z) = (r\delta s)\alpha x + (r\delta s)\alpha z = s\delta(r\alpha x) + r\delta(s\alpha z) \leq r\alpha x + s\alpha z = 0$. This implies that $t\alpha(x + z) = 0$. Similarly, $(x + z)\alpha t = 0$. Further, $t + y = t + 1\delta y = t + (r + 1)\delta y = r\delta s + r\delta y + 1\delta y = r\delta(s + y) + y = r\delta 1 + y = r + y = 1$. Hence, $(x + z)\nabla y$. □

Theorem 3.29. *Let R be a positive, additively idempotent and simple $\Gamma-$ semiring. If $y \in R$ then $I = \{z \in R \mid r\nabla y\}$ is an ideal of R.*

Proof. Let $x \in I, z \in R$ and $\alpha \in \Gamma$. Then $x\alpha z \leq x\nabla y$. So, by Theorem 3.26, $(z\alpha x)\nabla y$. Similarly, $(x\alpha z)\nabla y$. Thus, $x\alpha z, z\alpha x \in I$. Further, if $x, z\nabla y$, that is, $x\nabla y$ and $z\nabla y$ then by Theorem 3.28, $(x + z)\nabla y$. Hence, $x + z \in I$. □

REFERENCES

[1] Dutta, T. K. and Sardar, S. K. , *Semiprime ideals and irreducible ideals of a* $\Gamma-$*semiring*, Novi Sad J. Math., 30(2000), 97-108.

[2] Golan, J.S., *Semirings and their applications*, Kluwer Academic Publishers 1999.

[3] Hedayati, H. and Shum, K.P. , *An introduction to* $\Gamma-$ *semirings*, International Journal Of Algebra, 5(2011), 709-726.

[4] Rao, M.M.K. , *On Field* $\Gamma-$ *semiring and Complemented* $\Gamma-$ *semiring with Identity*, Bulletin Of The International Mathematical Virtual Institute, 8(2017), **85-99**.

[5] Rao, M.M.K. , $\Gamma-$ *semirings-I* , Southeast Asian Bull. of Math., 19(1995), 49-54.

[6] Nobusawa, N. , *On a generalization of the ring theory*, Osaka J. Math. 1(1964), 81-89.

[7] Sen, M. K. , $\Gamma-$ *Semigroup*, Proc. of Inter. Con. of alg. and its appl., Decker Publication, New York (1981), 301-308.

[8] Sharma, Tilak Raj , *Lattices in* $\Gamma-$ *Semirings*, International Journal of Creative Research Thoughts (IJCRT), 8(2020), 3360-3366.

[9] Sharma, Tilak Raj and Gupta, Shweta , *Some Conditions on* $\Gamma-$ *Semirings*, J. of Combinatorics Information and System Sciences, 41(2016), 79-87.

[10] Sharma, Tilak Raj and Gupta, Shweta, *Complementation of* $\Gamma-$ *semirings*, JP Journal of Algebra, Number theory and Applications, 50(2021), 101-112.

[11] Sharma, Tilak Raj and Ranote, Hitesh Kumar, *On Some Properties of a* $\Gamma-$*semiring*, JP Journal of Algebra, Number Theory and Applications, 52(2021), 163-177.

[12] Sharma, Tilak Raj and Ranote, Hitesh Kumar, *Gel' fand* $\Gamma-$ *semirings*, Journal of Mathematica Panonica (2022), 01-07.