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Generalized Lower Order and Generalized Lower Type of Entire Monogenic Functions of Slow Growth

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Abstract

The concept of generalized growth of “entire monogenic functions” with slow growth is introduced in this paper. Taylor’s series coefficients have been used to characterize the generalized lower type and lower order of “entire monogenic functions”.

Keywords: Generalized Cauchy-Riemann system, Clifford algebra, Clifford analysis, Generalized lower type, Generalized lower order, Entire monogenic function.

AMS Subject Classification: 30G35

1. INTRODUCTION

Clifford analysis provides the ability to generalize the theory of complex function in higher dimensions. It analyses the function of Clifford algebraic values which are described in an open subset of $\mathbb{R}^m$, $m \in \mathbb{N}$ and which are solutions of higher-dimensional Cauchy-Riemann systems. These are known as Clifford monogenic or holomorphic functions.

We use the following notations to induce the calculation more concisely:

For $\mathbf{h} = (h_1, \dots, h_m) \in \mathbb{N}_0^m$ and $\mathbf{z} \in (z_1, \dots, z_m) \in \mathbb{R}^m$,

$$\mathbf{z}^{\mathbf{h}} = z_1^{h_1} \dots z_m^{h_m} \ , \ \mathbf{h}! = h_1! \dots h_m! \ , \ |\mathbf{h}| = h_1 + \dots + h_m.$$

$\{r_1, r_2, \dots, r_m\}$ denotes the basis of $\mathbb{R}^m$. The related real Clifford algebra Cl_{0m} is the free algebra formed by $\mathbb{C}^m$ modulo $\mathbf{z}^2 = -\|\mathbf{z}\|^2 r_0$. Here r_0 is a neutral element concerning the multiplication of Clifford algebra Cl_{0m}. In the Clifford algebra Cl_{0m} following multiplication rule holds: $r_s r_e + r_e r_s = -2\alpha_{se} r_0$, $s, e = 1, 2, ..., m$, where α_{se} described as a Kronecker symbol.

For Clifford algebra Cl_{0m} basis is given by the set $\{r_B : B \subseteq \{1, 2, ..., m\}\}$ with $r_B = r_{l_1} r_{l_2} ... r_{l_c}$, where $1 \le l_1 < l_2 < \cdots < l_c \le m$, $r_\varphi = r_0 = 1$.

Each $b \in Cl_{0m}$ can be written as $b = \sum_B b_B r_B$ with $b_B \in \mathbb{C}$. In Clifford algebra Cl_{0m} the conjugation is defined as $\bar{b} = \sum_B b_B \bar{r}_B$, where $\overline{r_B} = \bar{r}_{l_c} \bar{r}_{l_{c-1}} ... \bar{r}_{l_1}$ and $\bar{r}_e = -r_e$ for

$e = 1, 2, ... m$, $\bar{r}_0 = r_0 = 1$.

The linear subspace

$$\operatorname{span}_C \{1, r_1, ..., r_m\} = \mathbb{C} \oplus \mathbb{C}^m \subset Cl_{0m}$$

is known as the space of para vectors $x = z_0 + z_1 r_1 + z_2 r_2 + ... + z_m r_m$ which we easily identify as $\mathbb{C}^{m+1}$.

Here $z_0 = \mathrm{Sc}(x)$ is the scalar part and $\mathbf{z} = z_1 r_1 + z_2 r_2 + ... + z_m r_m = \mathrm{Vec}(x)$ is vector part of para vector x.

$\| b \| = \left(\sum_B | b_B |^2 \right)^{1/2}$ is Clifford norm of an arbitrary $b = \sum_B b_B r_B$.

Every para vector $x \in \mathbb{C}^{m+1} \setminus \{0\}$ has an inverse element in $\mathbb{C}^{m+1}$ that can be written as $x^{-1} = \bar{x} / \| x \|^2$. The generalized Cauchy Riemann operator in $\mathbb{C}^{m+1}$ is given by

$$K \equiv \frac{\partial}{\partial z_0} + \sum_{s=1}^{m} r_s \frac{\partial}{\partial z_s} .$$

For the open set $A \subseteq \mathbb{C}^{m+1}$, a function $f : A \to Cl_{0m}$ is "left (right)" monogenic at the point $x \in A$, if $K\, f(x) = 0$ $(f\, K(x) = 0)$.

Let B_{m+1} be m-dimensional "surface area" of $m+1$-dimensional unit ball, and the Cauchy kernel function is $p_0(x) = \dfrac{\bar{x}}{\|x\|^{m+1}}$. Then each function f that is monogenic in the neighborhood of the domain $\overline{W}$ ($\overline{W}$ is the closure of W) satisfies the equation ([3], p. 766)

$$f(x)=\frac{1}{B_{m+1}}\int_{\partial W} p_{\mathbf{0}}(x-\pi)\ d\tau(\pi)\ f(\pi)\ ,\ \forall\ x\in W.$$

Here $d\tau(\pi)=\sum_{e=0}^{m}(-1)^{e} r_e\, d\pi_e$ and $d\pi_e=d\pi_0\wedge\ldots\wedge d\pi_{e-1}\wedge d\pi_{e+1}\wedge\ldots\wedge d\pi_m$ is the oriented outer normal "surface measure".

If f is a left monogenic function in a ball $\|x\|<C,$ then $\forall\ \|x\|<c$ with $0<c<C,$

$$f(x)=\sum_{|\mathbf{h}|=0}^{\infty} T_{\mathbf{h}}(x)b_{\mathbf{h}}. \tag{1.1}$$

In (1.1), $T_{\mathbf{h}}(x)$ is known as the Fueter polynomials and is given as

$$T_{\mathbf{h}}(x)=\mathbf{h!}\big/|\mathbf{h}|!\sum_{\kappa\in perm(\mathbf{h})} x_{\kappa(h_1)}\ldots x_{\kappa(h_m)},$$

where $perm(\mathbf{h})$ are all permutations set of the sequence $(h_1,h_2\ldots,h_m)$ and $x_s=z_s-z_0r_s$ for $s=1,\ldots,m$ and $T_{\mathbf{0}}(x)=1.$

Again in (1.1), Clifford numbers $\{b_{\mathbf{h}}\}$ is defined as

$$b_{\mathbf{h}}=\frac{1}{\mathbf{h!}B_{m+1}}\int_{\|\pi\|<c} p_{\mathbf{h}}(\pi)\ d\tau(\pi)\, f(\pi)$$

and satisfy the inequality

$$\|b_{\mathbf{h}}\|\le d(y,\mathbf{h})\frac{M(c,f)}{c^{|\mathbf{h}|}}.$$

Here $M(c,f)=\max_{\|x\|=c}\{\|f(x)\|\}$ represents the maximum modulus of the function f in a closed ball of radius c and

$$p_{\mathbf{h}}(x)=\frac{\partial^{h_0+h_1+\ldots+h_m}}{\partial z_0^{h_0}\partial z_1^{h_1}\ldots\partial z_m^{h_m}}\, p_{\mathbf{0}}(x),\ \ d(y,\mathbf{h})=\frac{y(y+1)\ldots(y+|\mathbf{h}|-1)}{\mathbf{h!}}.$$

The term "central index" and "maximum term" were first introduced by Almeida and Krausshar [1].

Suppose $f:\square^{m+1}\to Cl_{0m}$ be a "left entire monogenic function" with the Taylor's series expansion $f(x)=\sum_{|\mathbf{h}|=0}^{\infty} T_{\mathbf{h}}(x)b_{\mathbf{h}}$. Then, for $c>0$, the "maximum term" of this entire monogenic function is denoted by

$$\delta(c)=\delta(c,f)=\max_{|\mathbf{h}|\ge 0}\{\|b_{\mathbf{h}}\|\,c^{|\mathbf{h}|}\}.$$

We further introduce, the index $\mathbf{h}$ with maximal length $|\mathbf{h}|$ for which "maximum term" is gained is known as "central index" and is given by $\beta(c) = \beta(c, f) = \mathbf{h}.$

The definitions of lower type, type, lower order and order of an "entire monogenic function" were first introduced by Kumar [14], Abul-Ez and Almaida [12].

Definition 1.1: (Order and Lower order)

Suppose $f : \square^{m+1} \to Cl_{0m}$ be an "entire monogenic function" with Taylor's series expansion $f(x) = \sum_{|\mathbf{h}|=0}^{\infty} T_{\mathbf{h}}(x) b_{\mathbf{h}}$. Then

$$\omega = \lim_{c\to\infty} \sup \frac{\ln\ln M(c, f)}{\ln c}, \tag{1.2}$$

is known as order of the function $f(x)$.

We further introduce

$$\kappa = \lim_{c\to\infty} \inf \frac{\ln\ln M(c, f)}{\ln c}, \tag{1.3}$$

as the lower (inferior) order of $f(x)$.

Definition 1.2: (Type and Lower type)

For an entire monogenic function $f : \square^{m+1} \to Cl_{0m}$ of order $1 < \omega < \infty$ the type μ and lower type η of $f(x)$ are defined by

$$\mu = \lim_{c\to\infty} \sup \frac{\ln M(c, f)}{c^{\omega}} , \tag{1.4}$$

$$\eta = \lim_{c\to\infty} \inf \frac{\ln M(c, f)}{c^{\omega}} . \tag{1.5}$$

If $f(x)$ is zero order entire monogenic function, then the equations {(1.4) and (1.5)} have no meaning.

Therefore, the growth of "entire monogenic functions" can't be compared by limiting the previous concepts. In order to solve this issue, following V.G. Iyer [9], we've invented the idea of lower logarithmic order, logarithmic order for "entire monogenic functions" with slow growth.

The lower logarithmic order, logarithmic order of an "entire monogenic function" $f(x)$ of zero order are defined as

$$\omega_l = \limsup_{c\to\infty} \frac{\ln\ln M(c, f)}{\ln\ln c}, \qquad \kappa_l = \liminf_{c\to\infty} \frac{\ln\ln M(c, f)}{\ln\ln c}. \tag{1.6}$$

Also, for entire monogenic function $f(x)$ with $1 < \omega_l < \infty$, the lower logarithmic type η_l and logarithmic type μ_l of $f(x)$ are defined as

$$\mu_l = \limsup_{c\to\infty} \frac{\ln M(c, f)}{(\ln c)^{\omega_l}}, \qquad \eta_l = \liminf_{c\to\infty} \frac{\ln M(c, f)}{(\ln c)^{\omega_l}}. \tag{1.7}$$

Following Srivastava and Kumar {[10] and [11]}, Almeida and Abul-Ez [12], the lower logarithmic order and logarithmic order of an "entire monogenic function" $f(x)$ in term of central index and "maximum term" are defined as

$$(\omega_l)_1 = \limsup_{c\to\infty} \frac{\ln\ln \delta(c)}{\ln\ln c}, \qquad (\kappa_l)_1 = \liminf_{c\to\infty} \frac{\ln\ln \delta(c)}{\ln\ln c}, \tag{1.8}$$

and

$$(\omega_l)_2 = \limsup_{c\to\infty} \frac{\ln\left[|\beta(c)|\right]}{\ln\ln c}, \qquad (\kappa_l)_2 = \liminf_{c\to\infty} \frac{\ln\left[|\beta(c)|\right]}{\ln\ln c}. \tag{1.9}$$

Again, for an entire monogenic function $f(x)$ with $1 < \omega_l < \infty$, we define

$$(\mu_l)_1 = \limsup_{c\to\infty} \frac{\ln \delta(c)}{(\ln c)^{\omega_l}}, \qquad (\eta_l)_1 = \liminf_{c\to\infty} \frac{\ln \delta(c)}{(\ln c)^{\omega_l}} \tag{1.10}$$

and

$$(\mu_l)_2 = \limsup_{c\to\infty} \frac{|\beta(c)|}{(\ln c)^{\omega_l}}, \qquad (\eta_l)_2 = \liminf_{c\to\infty} \frac{|\beta(c)|}{(\ln c)^{\omega_l}}. \tag{1.11}$$

The concept of generalized type and generalized order for entire transcendental functions was given by Kapoor, Nautiyal [5] and Seremeta [8]. Suppose the class of functions $l(z)$ is represented by L^0 satisfying the following conditions:

(i) strictly increasing, positive, and differentiable function $l(z)$ is defined on $[b,\infty)$ and tends to ∞ as $x \to \infty$,

(ii) $$\lim_{z\to\infty} \frac{l[\{1+1/v(z)\}z]}{l(z)} = 1,$$

for every function $v(z)$ such that $v(z) \to \infty$ as $z \to \infty$.

Let Ω be the family of the functions $l(z)$ which satisfies the conditions (i) and

(iii) $$\lim_{c\to\infty} \frac{l(dz)}{l(z)} = 1,$$

for every $d > 0$, that is $l(z)$ is increasing slowly.

Suppose Λ be the family of the functions $l(z)$ which satisfies the conditions (i) and

(iv) $\exists$ a function $\beta(x) \in \Omega$, and constants z_0 , R_1 and R_2 such that

$$0 < R_1 \le \frac{d\{l(z)\}}{d\{\beta(\ln z)\}} \le R_2 < \infty \quad ,$$

for all $z > z_0$.

Suppose $\overline{\Lambda}$ be the family of the functions $l(z)$ which satisfies (i) and

(v) $$\lim_{z\to\infty} \frac{d\{l(z)\}}{d(\ln z)} = R \, , \quad 0 < R < \infty.$$

Classes Λ and $\overline{\Lambda}$ are contained in Ω and $\Lambda \cap \overline{\Lambda} = \phi$.

Following S. Kumar and K. Bala {[6] and [7]}, Srivastava and S. Kumar {[10] and [11]} and S. Kumar [14], we define generalized type, generalized order, generalized lower order and lower type of an “entire monogenic function” having slow-growth.

For an “entire monogenic function” $f(x)$ and a function $\gamma(z)$ either belongs to Λ or to $\overline{\Lambda}$, the generalized lower order $\kappa(\gamma, f)$ and generalized order $\omega(\gamma, f)$ of $f(x)$ are defined as

$$\omega(\gamma, f) = \limsup_{c\to\infty} \frac{\gamma[\ln M(c, f)]}{\gamma(\ln c)}, \qquad \kappa(\gamma, f) = \liminf_{c\to\infty} \frac{\gamma[\ln M(c, f)]}{\gamma(\ln c)}. \qquad (1.12)$$

If we substitute $\gamma(c) = \ln c$ into the above equation, then we obtain the definition as given in (1.6). Again, for $\gamma(z) \in L^0$, we define generalized lower type $\eta(\gamma, \omega, f)$ and generalized type $\mu(\gamma, \omega, f)$ of entire monogenic function $f(x)$ with $\omega = \omega(\gamma, f)(1 < \omega < \infty)$ are defined as

$$\mu(\gamma, \omega, f) = \limsup_{c\to\infty} \frac{\gamma\{\ln M(c, f)\}}{\{\gamma(\ln c)\}^{\omega}} \ , \quad \eta(\gamma, \omega, f) = \liminf_{c\to\infty} \frac{\gamma\{\ln M(c, f)\}}{\{\gamma(\ln c)\}^{\omega}} \ . \qquad (1.13)$$

If we substitute $\gamma(c) = c$ into the above equation, then we obtain definition as given in (1.7). We further introduced

for, an entire monogenic function $f(x)$ and a function $\gamma(z)$ that either belongs to Λ or to $\overline{\Lambda}$, in term of central index and “maximum term”, we define the generalized lower order, generalized order of $f\left(x\right)$ as

$$\omega_1(\gamma,f)=\lim_{c\to\infty}\sup\frac{\gamma\left[\ln\delta\left(c\right)\right]}{\gamma(\ln c)},\qquad \kappa_1(\gamma,f)=\lim_{c\to\infty}\inf\frac{\gamma\left[\ln\delta(c)\right]}{\gamma(\ln c)}$$

and

$$\omega_2(\gamma,f)=\lim_{c\to\infty}\sup\frac{\gamma\left[\left|\beta(c)\right|\right]}{\gamma(\ln c)},\qquad \kappa_2(\gamma,f)=\lim_{c\to\infty}\inf\frac{\gamma\left[\left|\beta(c)\right|\right]}{\gamma(\ln c)}. \tag{1.14}$$

If we substitute $\gamma\left(c\right)=\ln c$ into the above equation, then we obtain definition as given in (1.8) and (1.9). Following Srivastava and Kumar {[13], p.673}, Abul-Ez and Almeida {[12], p. 1263}, we have

and

$$\begin{gathered}\kappa\left(\gamma,f\right)\le\kappa_1(\gamma,f)=\kappa_2\left(\gamma,f\right),\\ \omega\left(\gamma,f\right)\le\omega_1(\gamma,f)=\omega_2\left(\gamma,f\right).\end{gathered} \tag{1.15}$$

Also, for an "entire monogenic function" $f(x)$ with $\omega=\omega(\gamma,f)\left(1<\omega<\infty\right)$ and $\gamma(x)\in L^0$, we define

$$\mu_1(\gamma,\omega,f)=\lim_{c\to\infty}\sup\frac{\gamma\left[\ln\delta(c)\right]}{\left[\gamma(\ln c)\right]^{\omega}},\qquad \eta_1(\gamma,\omega,f)=\lim_{c\to\infty}\inf\frac{\gamma\left[\ln\delta(c)\right]}{\left[\gamma(\ln c)\right]^{\omega}}$$

and

$$\mu_2(\gamma,\omega,f)=\lim_{c\to\infty}\sup\frac{\gamma\left[\left|\beta(c)\right|\right]}{\left[\gamma(\ln c)\right]^{\omega}},\qquad \eta_2(\gamma,\omega,f)=\lim_{c\to\infty}\inf\frac{\gamma\left[\left|\beta(c)\right|\right]}{\left[\gamma(\ln c)\right]^{\omega}}.$$

If we substitute $\gamma\left(c\right)=c$ into the above equation, then we obtain definition as given in (1.10) and (1.11).

2. Main Results

"Now we prove"

Theorem 2.1: Suppose $f:\square^{m+1}\to Cl_{0_m}$ is an entire monogenic function with Taylor's series expansion $f(x)=\sum_{|\mathbf{h}|=0}^{\infty}b_{\mathbf{h}}\,T_{\mathbf{h}}(x)$.

Also, if $\gamma(z)$ either belongs to Λ or to $\overline{\Lambda}$, then the generalized lower order

$\kappa(\gamma, f)\left(1<\kappa(\gamma, f)<\infty\right)$ of this “entire monogenic function” $f(x)$ satisfies

$$\kappa(\gamma, f)-1 \geq \liminf_{|\mathbf{h}|\to\infty} \frac{\gamma\left(|\mathbf{h}|\right)}{\gamma\left\{\ln\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|^{-1/|\mathbf{h}|}\right\}}. \tag{2.1}$$

Further, if $\psi(r)=\max\limits_{|\mathbf{h}|=r}\left\{\dfrac{\|b_{\mathbf{h}}\|}{\|b_{\mathbf{h}'}\|},\ \|\mathbf{h}'\|=\|\mathbf{h}\|+1\right\}$ is a non-decreasing function of r, then equality holds in (2.1).

Proof:

Write $\kappa=\kappa(\gamma, f)$ and

$$\Phi=\liminf_{|\mathbf{h}|\to\infty} \frac{\gamma\left(|\mathbf{h}|\right)}{\gamma\left\{\ln\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|^{-1/|\mathbf{h}|}\right\}}.$$

First, we show that $\Phi\leq\kappa-1$.

Using Cauchy’s inequality, we have

$$\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\| \leq c^{-|\mathbf{h}|}M(c, f). \tag{2.2}$$

Again, from (1.12) for arbitrary $\varepsilon>0$, and a sequence $c=c_u\to\infty$ as $u\to\infty$, we have

$$M(c, f)\leq\exp\left[\gamma^{-1}\left\{\bar{\kappa}\gamma(\ln c)\right\}\right],$$

where $\bar{\kappa}=\kappa+\varepsilon$.

Now from (2.2), we get

$$\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\| \leq c^{-|\mathbf{h}|}\exp\left[\gamma^{-1}\left\{\bar{\kappa}\gamma(\ln c)\right\}\right]$$

or

$$\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\| \leq \exp\left[-|\mathbf{h}|\ln c+\gamma^{-1}\left\{\bar{\kappa}\gamma(\ln c)\right\}\right]. \tag{2.3}$$

Let $c=c\left(|\mathbf{h}|\right)$ be the root of the equation, because $\gamma(z)$ is an increasing function of (z).

$$\gamma\left[\frac{|\mathbf{h}|\ln c}{\bar{\kappa}}\right]=\bar{\kappa}\gamma(\ln c). \tag{2.4}$$

Then, for suitably large values of $|\mathbf{h}|$, we have

$$\ln c \,\square\, \gamma^{-1}\left[\frac{1}{\bar{\kappa}-1}\gamma\left(|\mathbf{h}|\right)\right]=Q\left(|\mathbf{h}|,\frac{1}{\bar{\kappa}-1}\right). \tag{2.5}$$

Here $Q\left(z,y\right)=\gamma^{-1}\left[y\gamma(z)\right]\ \forall\ z,y\in C.$

Using (2.4) and (2.5) in (2.3), we obtain

$$\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|\le\exp\left[-|\mathbf{h}|Q+\left(|\mathbf{h}|/\bar{\kappa}\right)Q\right]$$

or

$$\frac{\bar{\kappa}}{\bar{\kappa}-1}\ln\left\{\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|\right\}^{-1/|\mathbf{h}|}\ge\gamma^{-1}\left[\frac{1}{\bar{\kappa}-1}\gamma\left(|\mathbf{h}|\right)\right]$$

or

$$\frac{\gamma\left(|\mathbf{h}|\right)}{\gamma\left\{\dfrac{\bar{\kappa}}{\bar{\kappa}-1}\ln\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|^{-1/|\mathbf{h}|}\right\}}\le\bar{\kappa}-1$$

or

$$\frac{\gamma\left(|\mathbf{h}|\right)}{\gamma\left\{\ln\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|^{-1/|\mathbf{h}|}\right\}}\le\left(\bar{\kappa}-1\right)\times$$

$$\times\frac{\gamma\left\{\dfrac{\bar{\kappa}}{\bar{\kappa}-1}\ln\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|^{-1/|\mathbf{h}|}\right\}}{\gamma\left\{\ln\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|^{-1/|\mathbf{h}|}\right\}}.$$

Since $\gamma(dz)\approx\gamma(z)$ as $z\to\infty.$ Taking limits as $|\mathbf{h}|=|\mathbf{h}(r)|\to\infty,$ we get

$$\Phi\le\bar{\kappa}-1.$$

Since $\varepsilon>0$ is arbitrarily small, at last, we get

$$\Phi\le\kappa-1.$$

Now, we show that $\kappa-1\le\Phi.$ From the assumption on $\psi,\ \psi(r)\to\infty$ as $r\to\infty.$

By the definition given in section 1, if $\left\|b_{\mathbf{h}}\right\|c^{|\mathbf{h}|}$ is the maximum term for c then for $|\mathbf{h}_1|\le|\mathbf{h}|<|\mathbf{h}_2|,$

$$\left\|b_{\mathbf{h}_1}\right\|c^{|\mathbf{h}_1|}\le\left\|b_{\mathbf{h}}\right\|c^{|\mathbf{h}|}<\left\|b_{\mathbf{h}_2}\right\|c^{|\mathbf{h}_2|}$$

and for $|\mathbf{h}|=r,$

$$\psi(r-1)\le c<\psi(r).$$

Now suppose that $\left\|b_{\mathbf{h}^1}\right\|c^{|\mathbf{h}^1|}$ and $\left\|b_{\mathbf{h}^2}\right\|c^{|\mathbf{h}^2|},$ are two consecutive maximum terms. Then

$$|\mathbf{h}^1| \le |\mathbf{h}^2| - 1.$$

Suppose

$$|\mathbf{h}^1| \le |\mathbf{h}| \le |\mathbf{h}^2|.$$

Then

$$|\beta(c)| = |\mathbf{h}^1|$$

for

$$\psi\left(|\mathbf{h}^{1^*}|\right) \le c < \psi\left(|\mathbf{h}^1|\right),$$

where $|\mathbf{h}^{1^*}| = |\mathbf{h}^1| - 1.$

Therefore, from (1.12) for arbitrary $\varepsilon > 0$ and all $c > c_0(\varepsilon),$ we have

$$|\mathbf{h}^1| = |\beta(c)| > \gamma^{-1}\left\{\kappa^{'}\gamma(\ln c)\right\} \;, \kappa^{'} = \kappa - \varepsilon$$

or

$$|\mathbf{h}^1| = |\,\beta(c)\,| \ge \gamma^{-1}\left\{\kappa^{'}\gamma\left[\ln\left\{\psi\left(|\mathbf{h}^1|\right) - w\right\}\right]\right\}.$$

Here w is a constant, such that

$$0 < w < \min\left\{1, \left[\psi\left(|\mathbf{h}^1|\right) - \psi\left(|\mathbf{h}^{1^*}|\right)\right]\Big/2\right\}$$

or

$$\ln\psi\left(|\mathbf{h}^1|\right) \le O(1) + \gamma^{-1}\left\{\gamma\left(|\mathbf{h}^1|\right)\Big/\kappa^{'}\right\}.$$

Further, we have

$$\psi\left(|\mathbf{h}^1|\right) = \psi\left(|\mathbf{h}^1| + 1\right) = \cdots = \psi\left(|\mathbf{h}| - 1\right).$$

Now we can write

$$\psi\left(|\mathbf{h}^0|\right)\cdots\,\psi\left(|\mathbf{h}^*|\right) = \frac{\|b_{\mathbf{h}^0}\|}{\|b_{\mathbf{h}}\|} \le \left[\psi\left(|\mathbf{h}^*|\right)\right]^{|\mathbf{h}| - |\mathbf{h}^0|}.$$

Here $|\mathbf{h}^*| = |\mathbf{h}| - 1$ and $|\mathbf{h}| \square |\mathbf{h}^0|$.

Hence

$$d(y,\mathbf{h})\frac{\left\|b_{\mathbf{h}^0}\right\|}{\left\|b_{\mathbf{h}}\right\|} \le d(y,\mathbf{h})\left[\psi(\left(\left|\mathbf{h}^*\right|\right))\right]^{|\mathbf{h}|-|\mathbf{h}^0|}$$

or

$$\ln\left\|b_{\mathbf{h}} / d(y,\mathbf{h})\right\|^{-1} \le |\mathbf{h}|\ln\psi\left(\left|\mathbf{h}^1\right|\right)+O(1)$$

$$\le |\mathbf{h}|\frac{1}{\gamma}\left\{\frac{\gamma\left(\left|\mathbf{h}^1\right|\right)}{\kappa^{'}}\right\}+O(1)$$

or

$$-\frac{1}{|\mathbf{h}|}\ln\left\|b_{\mathbf{h}} / d(y,\mathbf{h})\right\| \le \left(\frac{1}{\gamma}\left\{\frac{\gamma\left(\left|\mathbf{h}^1\right|\right)}{\kappa^{'}}\right\}\right)[1+o(1)]$$

$$\le \left[\frac{1}{\gamma}\left\{\frac{\gamma\left(\left|\mathbf{h}^1\right|\right)}{\kappa^{'}}\right\}\right][1+o(1)]$$

or

$$\kappa^{'} \le \frac{\gamma\left(|\mathbf{h}|\right)}{\gamma\left\{\ln\left\|b_{\mathbf{h}} / d(y,\mathbf{h})\right\|^{-1/|\mathbf{h}|}\right\}}[1+o(1)].$$

Taking the limit as $|\mathbf{h}|\to\infty,$ we obtain

$$\kappa\le\Phi$$

or

$$\kappa-1\le\Phi.$$

The proof is completed.

“Next we prove”

Theorem 2.2

Suppose $f:\mathbb{R}^{m+1}\to Cl_{0m}$ is an “entire monogenic function” with Taylor’s series expansion $f(x)=\sum_{|\mathbf{h}|=0}^{\infty}b_{\mathbf{h}}T_{\mathbf{h}}(x)$. Then for $\gamma(z)\in L^0$ and $\omega=\omega(\gamma,f)(1<\omega<\infty)$ the generalized lower type $\eta(\gamma,\ \omega,\ f)$ of $f(x)$ is given as

$$\eta(\gamma,\omega,f) \geq \liminf_{|\mathbf{h}|\to\infty} \frac{\gamma\left(|\mathbf{h}|/\omega\right)}{\left(\gamma\left\{\frac{\omega}{\omega-1}\ln\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|^{-1/|\mathbf{h}|}\right\}\right)^{\omega-1}}. \tag{2.6}$$

Further, if $\psi(r) = \max_{|\mathbf{h}|=r}\left\{\frac{\|b_{\mathbf{h}}\|}{\|b_{\mathbf{h}'}\|},\ \|\mathbf{h}'\| = \|\mathbf{h}\|+1\right\}$ is a non-decreasing function of r, then equality holds in (2.6).

Proof:

Write $\eta = \eta\left(\gamma,\omega,f\right)$ and

$$\zeta = \liminf_{|\mathbf{h}|\to\infty} \frac{\gamma\left(|\mathbf{h}|/\omega\right)}{\left(\gamma\left\{\omega/_{\omega-1}\ln\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|^{-1/|\mathbf{h}|}\right\}\right)^{\omega-1}}.$$

Again from (1.13), for arbitrary $\varepsilon>0$, and sequence $c = c_u \to \infty$ as $u \to \infty$, we have

$$M(c,f) \leq \exp\left[\gamma^{-1}\left\{\bar{\eta}\left[\gamma(\ln c)\right]^{\omega}\right\}\right], \tag{2.7}$$

where $\bar{\eta} = \eta + \varepsilon$.

Now from (2.2), we obtain

$$\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\| \leq c^{-|\mathbf{h}|}\exp\left[\gamma^{-1}\left\{\bar{\eta}\left[\gamma(\ln c)\right]^{\omega}\right\}\right]$$

or

$$\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\| \leq \exp\left[-|\mathbf{h}|\ln c + \gamma^{-1}\left\{\bar{\eta}\left[\gamma(\ln c)\right]^{\omega}\right\}\right]. \tag{2.8}$$

Let $c = c\left(|\mathbf{h}|\right)$ be the root of the equation, because $\gamma(z)$ is an increasing function of (z).

$$\gamma\left[\frac{|\mathbf{h}|\ln c}{\omega}\right] = \bar{\eta}\left[\gamma(\ln c)\right]^{\omega}. \tag{2.9}$$

Then, for suitably large values of $|\mathbf{h}|$, we have

$$\ln c \square \gamma^{-1}\left[\left(\frac{1}{\bar{\eta}}\gamma\left(|\mathbf{h}|/\omega\right)\right)^{1/1-\omega}\right] = Z\left(|\mathbf{h}|/\omega, \frac{1}{\bar{\eta}}, \omega-1\right). \tag{2.10}$$

Using (2.9) and (2.10) in (2.8), we get

$$\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\| \leq \exp\left[-|\mathbf{h}|Z + \left(|\mathbf{h}|/\omega\right)Z\right]$$

or

$$\frac{\omega}{\omega-1}\ln\left\{\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|\right\}^{-1/|\mathbf{h}|} \geq \gamma^{-1}\left[\left\{\frac{1}{\bar{\eta}}\gamma\left(|\mathbf{h}|/\omega\right)\right\}^{1/\omega-1}\right]$$

or

$$\frac{\gamma\left(|\mathbf{h}|/\omega\right)}{\left[\gamma\left\{\frac{\omega}{\omega-1}\ln\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|^{-1/|\mathbf{h}|}\right\}\right]^{\omega-1}} \leq \bar{\eta}.$$

Taking limits as $|\mathbf{h}| = |\mathbf{h}(r)| \to \infty,$ we obtain

$$\zeta \leq \bar{\eta}.$$

Since $\varepsilon > 0$ is arbitrarily small, at last we obtain

$$\zeta \leq \eta. \tag{2.11}$$

Now, we have to show that $\zeta \leq \eta$. If $\zeta = \infty$, then nothing to show here. So let us suppose that $0 \leq \zeta < \infty.$ Therefore, for a given $\varepsilon > 0,$ $\exists$ $y_0 \in$□ such that $\forall$ multi-indices $\mathbf{h}$ with $|\mathbf{h}| > n_0,$ we have

$$0 \leq \frac{\gamma\left(|\mathbf{h}|/\omega\right)}{\left[\gamma\left\{\omega/_{\omega-1}\ln\left\|b_{\mathbf{h}}/d(y,\mathbf{h})\right\|^{-1/|\mathbf{h}|}\right\}\right]^{\omega-1}} < \zeta + \varepsilon = \bar{\zeta}$$

or

$$\|b_{\mathbf{h}}\| \leq d(y,\mathbf{h})\exp\left(1-\omega/_{\omega}|\mathbf{h}|\gamma^{-1}\left[\left\{1/_{\bar{\zeta}}\gamma\left(|\mathbf{h}|/\omega\right)\right\}^{1/_{1-\omega}}\right]\right).$$

By the maximum modulus property, we have

$$M(c,f) \leq \sum_{|\mathbf{h}|=0}^{\infty}\|b_{\mathbf{h}}\|\, c^{|\mathbf{h}|}$$

or

$$M(c,f) \leq \sum_{|\mathbf{h}|=0}^{y_0}\|b_{\mathbf{h}}\|\, c^{|\mathbf{h}|} + \sum_{|\mathbf{h}|=y_0+1}^{\infty} d(y,\mathbf{h})\, c^{|\mathbf{h}|}\exp\left(1-\omega/_{\omega}|\mathbf{h}|\gamma^{-1}\left[\left\{\frac{1}{\bar{\zeta}}\gamma\left(|\mathbf{h}|/\omega\right)\right\}^{1/_{1-\omega}}\right]\right).$$

Now for $c > 1,$ we have

$$M(c,f)\ \le\ A_1 c^{y_0}+\sum_{|\mathbf{h}|=y_0+1}^{\infty} d(y,\mathbf{h})\,c^{|\mathbf{h}|}\,\exp\left(\frac{1-\omega}{\omega}|\mathbf{h}|\gamma^{-1}\left[\left\{\frac{1}{\zeta}\,\gamma\left(|\mathbf{h}|/\omega\right)\right\}^{\frac{1}{1-\omega}}\right]\right), \tag{2.12}$$

A_1 represents, a positive real constant. We choose

$$X(c)=\left[\omega\gamma^{-1}\left(\overline{\zeta}\left[\gamma\left\{\frac{\omega}{\omega-1}\ln\left[(y+1)c\right]\right\}\right]^{\omega-1}\right)\right],$$

where $[z]$ represents the integer portion of $z\ge 0$. Since $\delta(z)\in L^0$, the integer $X(c)$ is properly defined. Now, if c is suitably large, then by (2.12), we have

$$\begin{aligned}M(c,f)\ \le\ & A_1 c^{y_0}+\\ &+c^{X(c)}\sum_{y_0+1\le|\mathbf{h}|\le X(c)} d(y,\mathbf{h})\exp\left(\frac{1-\omega}{\omega}|\mathbf{h}|\gamma^{-1}\left[\left\{\frac{1}{\zeta}\,\gamma\left(|\mathbf{h}|/\omega\right)\right\}^{\frac{1}{1-\omega}}\right]\right)+\\ &+\sum_{|\mathbf{h}|>X(c)} d(y,\mathbf{h})\,c^{|\mathbf{h}|}\,\exp\left(\frac{1-\omega}{\omega}|\mathbf{h}|\gamma^{-1}\left[\left\{\frac{1}{\zeta}\,\gamma\left(|\mathbf{h}|/\omega\right)\right\}^{\frac{1}{1-\omega}}\right]\right)\end{aligned}$$

or

$$\begin{aligned}M(c,f)\ \le\ & A_1 c^{y_0}+\\ &+c^{X(c)}\sum_{|\mathbf{h}|=1}^{\infty} d(y,\mathbf{h})\,\exp\left(\frac{1-\omega}{\omega}|\mathbf{h}|\gamma^{-1}\left[\left\{\frac{1}{\zeta}\,\gamma\left(|\mathbf{h}|/\omega\right)\right\}^{\frac{1}{1-\omega}}\right]\right)+\\ &+\sum_{|\mathbf{h}|>X(c)} d(y,\mathbf{h})\,c^{|\mathbf{h}|}\,\exp\left(\frac{1-\omega}{\omega}|\mathbf{h}|\gamma^{-1}\left[\left\{\frac{1}{\zeta}\,\gamma\left(|\mathbf{h}|/\omega\right)\right\}^{\frac{1}{1-\omega}}\right]\right).\end{aligned} \tag{2.13}$$

Now, we can rewrite the first series in (2.13) as

$$\sum_{q=1}^{\infty}\left(\sum_{|\mathbf{h}|=q} d(y,\mathbf{h})\right)\exp\exp\left(\frac{1-\omega}{\omega}\,q\gamma^{-1}\left[\left\{\frac{1}{\zeta}\,\gamma\left(q/\omega\right)\right\}^{\frac{1}{1-\omega}}\right]\right). \tag{2.14}$$

Now from ([4], Lemma 1), we have

$$\liminf_{q\to\infty}\left(\sum_{|\mathbf{h}|=q} d(y,\mathbf{h})\right)^{\frac{1}{q}}=y.$$

Hence, we have

$$\liminf_{q\to\infty}\left[\left(\sum_{|\mathbf{h}|=q} d(y,\mathbf{h})\right)\ \exp\left(1-\omega\big/_{\omega}\, q\gamma^{-1}\left[\left\{1\big/_{\zeta}\,\gamma(q/\omega)\right\}^{1/1-\omega}\right]\right)\right]^{1/q}$$

$$= y\liminf_{q\to\infty}\exp\left(1-\omega\big/_{\omega}\, q\gamma^{-1}\left[\left\{1\big/_{\zeta}\,\gamma(q/\omega)\right\}^{1/1-\omega}\right]\right) = \ 0.$$

Therefore, in (2.14), the series converges to A_2 (where A_2 is positive real constant). Thus from (2.13), we obtain

$$M(c,f) \le A_1 c^{y_0} + A_2 c^{X(c)} +$$
$$+\sum_{|\mathbf{h}|>X(c)} d(y,\mathbf{h})\ c^{|\mathbf{h}|}\ \exp\left(1-\omega\big/_{\omega}\ |\mathbf{h}|\ \gamma^{-1}\left[\left\{\frac{1}{\zeta}\gamma\left(|\mathbf{h}|\big/_{\omega}\right)\right\}^{1/\omega-1}\right]\right)$$

or

$$M(c,f) \le\ A_1 c^{y_0} + A_2 c^{X(c)} + \sum_{|\mathbf{h}|>N(c)}^{\infty} d(y,\mathbf{h}) c^{|\mathbf{h}|} \exp\left[-|\mathbf{h}|\ln\left\{[y+1]c\right\}\right]$$

or

$$M(c,f) \le\ A_1 c^{y_0} + A_2 c^{X(c)} + \sum_{|\mathbf{h}|=1}^{\infty} d(y,\mathbf{h}) c^{|\mathbf{h}|} \exp\left[-|\mathbf{h}|\ln\left\{[y+1]c\right\}\right]$$

or

$$M(c,f) \le\ A_1 c^{y_0} + A_2 c^{X(c)} + \sum_{|\mathbf{h}|=1}^{\infty} d(y,\mathbf{h})\left(\frac{1}{y+1}\right)^{|\mathbf{h}|}. \tag{2.15}$$

The series in (2.15) can be rewritten as

$$\sum_{q=1}^{\infty}\left(\sum_{|\mathbf{h}|=q} d(y,\mathbf{h})\right)\left(\frac{1}{y+1}\right)^{q}. \tag{2.16}$$

So, we have

$$\liminf_{q\to\infty}\left[\sum_{q=1}^{\infty}\left(\sum_{|\mathbf{h}|=q} d(y,\mathbf{h})\right)\left(\frac{1}{y+1}\right)^{q}\right]^{1/q} \quad = \frac{y}{y+1} < 1.$$

The series in (2.16), converges to A_3 (where A_3 is positive real constant). Thus from (2.15), we obtain

$$M(c,f) \le A_1 c^{y_0} + A_2 c^{X(c)} + A_3.$$

Since $X(c) \to \infty$ as $c \to \infty$, this inequality can be rewrite as

$$\ln M(c,f) \le \left[1+o(1)\right] X(c) \ln c$$

or

$$\begin{aligned}\ln M(c,f) \le\ & [1+o(1)]\times \\ & \times\left[\omega\gamma^{-1}\left(\bar{\zeta}\left\{\gamma\left[{}^{\omega}\!/_{\omega-1}\ \ln\left\{(y+1)c\right\}\right]\right\}^{\omega-1}\right)\right]\ln c\end{aligned}$$

or

$$\begin{aligned}\ln M(c,f) \le [1+o(1)]\left[\omega\gamma^{-1}\left(\bar{\zeta}\left\{\gamma\left[{}^{\omega}\!/_{\omega-1}\ \ln\left\{(y+1)c\right\}\right]\right\}^{\omega-1}\right)\right]\times \\ \times\gamma^{-1}\left\{\gamma\left[{}^{\omega}\!/_{\omega-1}\ \ln\left\{(y+1)c\right\}\right]\right\}.\end{aligned}$$

Now, for suitably large values of c, we obtain using the properties of γ.

$$\gamma\left[\ln M(c,f)\right] \le [1+o(1)]\bar{\zeta}\left\{\gamma\left[{}^{\omega}\!/_{\omega-1}\ln\left\{(y+1)c\right\}\right]\right\}^{\omega}$$

or

$$\frac{\gamma\left\{\ln M(c,f)\right\}}{\left\{\gamma\left(\ln c\right)\right\}^{\omega}} \le [1+o(1)]\bar{\zeta}\left\{\frac{\gamma\left[{}^{\omega}\!/_{\omega-1}\ln\left\{(y+1)c\right\}\right]}{\gamma\left(\ln c\right)}\right\}^{\omega}.$$

Taking the limit as $c \to \infty$, and using properties of γ, we obtain

$$\eta \le \bar{\zeta}\ .$$

Since $\varepsilon > 0$ is arbitrarily small, at last, we obtain

$$\eta \le \zeta. \tag{2.17}$$

From (2.11) and (2.17), we obtain (2.6).

The proof is completed.

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