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On the Projective Flatness of a Special Finsler Space

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Abstract

In the current paper, we investigate the properties of special class of (α, β)-metric $F = \alpha + \beta + \beta\, tan^{-1}(\beta/\alpha)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. This has its application in theoretical physics. We obtain a necessary and sufficient condition for this metric to be locally projectively flat.

Keywords: Finsler space, $(\alpha$, $\beta)$-metric, Projectively flat Finsler space.

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1. INTRODUCTION

In the theory of Hilbert's Fourth Problem [9], the author constructs metrics on open subset in R^n such that straight lines are geodesics . In [[2]], Busemann calls these metrics as projective metrics. Finsler metric F are projectively flat if and only if the geodesic coefficients of F are in the following form

$$G^i = Py^i.$$

where $P = \dfrac{F_{x^k} y^k}{2F}$. The scalar function P is called the projective factor of F.
Projectively flat Finsler metrics must be a scalar flag curvature. It is interesting to study some important projectively flat Finsler metrics such as Riemannian metrics, Rander metrics [12], Matsumoto metrics [1] etc. A Riemannian metric [4] is projectively flat if and only if it is of constant sectional curvature. A Randers metric $F = \alpha + \beta$ is

projectively flat [12]if and only if α is of constant sectional curvature (i.e. projectively flat) and β is closed. Shen and Yidirim [13] studied the locally projectively flat metric in the form $F = \dfrac{(\alpha+\beta)^2}{\alpha}$.
The main purpose of this paper is to investigate the locally projective flat for (α, β)-metric $F = \alpha + \beta + \beta\, tan^{-1}(\beta/\alpha)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ is 1-form.
Now we have the following theorem :

Theorem 1.1. *The (α, β)-metric $F = \alpha + \beta + \beta\, tan^{-1}(\beta/\alpha)$ is a locally projective flat iff*
(i) β is parallel with respect to α,
(ii) α is locally projectively flat, i.e., of constant curvature.

2. PRELIMINARIES

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on an n-dimensional manifold M. An (α, β)-metric is a Finsler metric that can be expressed

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a continuous differentiable function on an open interval $(-b_0, b_0)$, satisfying

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \qquad |\, s \,| \leq b < b_0$$

where β satisfies $\| \, \beta_x \, \|_\alpha < b_0$, for any x $\in$ M.
Let G^i and G^i_α denote the spray coefficients of F and α respectively, given by

$$G^i = \frac{g^{il}}{4}\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\}, \quad G^i_\alpha = \frac{a^{il}}{4}\{[\alpha^2]_{x^k y^l}y^k - [\alpha^2]_{x^l}\},$$

where $g_{ij} := \dfrac{1}{2}[F^2]_{y^i y^j}$ and $(a^{ij}) := (a_{ij})^{-1}$. We have the following

Lemma 2.1. *The spray coefficients G^i are related to G^i_α by [[8]]*

$$G^i = G^i_\alpha + \alpha Q s^i_0 + J(-2Q\alpha s_0 + r_{00})\frac{y^i}{\alpha} + H(-2Q\alpha s_0 + r_{00})\Big\{b^i - s\frac{y^i}{\alpha}\Big\}, \qquad (1)$$

where

$$Q := \frac{\phi^{'}}{\phi - s\phi^{'}},$$

$$J := \frac{\phi^{'}(\phi - s\phi^{'})}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')},$$

$$H := \frac{\phi^{''}}{2((\phi - s\phi') + (b^2 - s^2)\phi'')},$$

$$s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),\ s_{l0} = s_{li}y^i,\ s_0 = s_{l0}b^l,\ r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}),\ and\, r_{00} = r_{ij}y^iy^j.$$

In 1903, G. Hamel [[5]] states that a Finsler metric F = F(x, y) on an open subset $U \subset R^n$ is projectively flat iff

$$F_{x^k y^l}y^k - F_{x^l} = 0. \tag{2}$$

By (2), we have the following lemma [13]:

Lemma 2.2. *An* (α, β)*-metric* $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, *is a projectively flat on an open subset* $U \subset R^n$ *iff*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0. \tag{3}$$

3. PROJECTIVELY FLAT (α, β)-METRIC

In this section, we consider the Finsler space with special (α, β)-metric $F = \alpha + \beta + \beta\, tan^{-1}(\beta/\alpha)$ that is obtained by [8]

$$F = \alpha\phi(s), \quad \phi(s) = 1 + s + s\, tan^{-1}s, \quad s = \beta/\alpha, \tag{4}$$

Let $b_0 > 0$ be the largest number such that

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \qquad |\, s\,| \leq b < b_0.$$

i.e.

$$\frac{1 + 2b^2 - s^2}{(1 + s^2)^2} > 0, \qquad |\, s\,| \leq b < b_0.$$

Lemma 3.1. $F = \alpha + \beta + \beta\, tan^{-1}(\beta/\alpha)$ *is a Finsler metric if and only if* $\|\ \beta_x\ \|_\alpha < 1$.
Proof: If $F = \alpha + \beta + \beta\, tan^{-1}(\beta/\alpha)$ *is a Finsler metric, then*

$$\frac{1 + 2b^2 - s^2}{(1 + s^2)^2} > 0, \qquad |\, s\,| \leq b < b_0.$$

Let s = b, then we get $b < 1,\ \forall\, b < b_0$. *Let* $b \to b_0$, *then* $b_0 < 1$. *So* $\|\ \beta_x\ \|_\alpha < 1$.
Conversely, if $|\, s\,| \leq b < 1$, *then*

$$\frac{1 + 2b^2 - s^2}{(1 + s^2)^2} \geq \frac{1 + s^2}{(1 + s^2)^2} > \frac{1}{1 + s^2} > 0.$$

Thus $F = \alpha + \beta + \beta\, tan^{-1}(\beta/\alpha)$ *is a Finsler metric.*

By Lemma 2.1, the Spray coefficients G^i of F are given by (1) with

$$Q = (1+s^2)(1+tan^{-1}s)+s = \frac{(\alpha^2+\beta^2)(1+tan^{-1}(\beta/\alpha))+\alpha\beta}{\alpha^2},$$

$$J = \frac{(1+s^2)(1+tan^{-1}s)+s}{2(1+\ s+s\,tan^{-1}s)(1+2b^2-s^2)} = \frac{\alpha(\alpha^2+\beta^2)(1+tan^{-1}(\beta/\alpha))+\alpha^2\beta}{2(\alpha+\beta+\beta\,tan^{-1}(\beta/\alpha))(\alpha^2(1+2b^2)-\beta^2)},$$

$$H = \frac{1}{1+2b^2-s^2} = \frac{\alpha^2}{\alpha^2(1+2b^2)-\beta^2}.$$

Equation (3) is reduced to the following equation:

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha((\alpha^2+\beta^2)(\epsilon+tan^{-1}(\beta/\alpha))+\alpha\beta)s_{l0}+$$

$$\frac{\alpha^3}{\alpha^2(1+2b^2)-\beta^2}\Big[\frac{-2\{(\alpha^2+\beta^2)(\epsilon+tan^{-1}(\beta/\alpha))+\alpha\beta\}}{\alpha}s_0+r_{00}\Big](b_l\alpha-\frac{\beta}{\alpha}y_l)=0. \quad (5)$$

Lemma 3.2. *If $(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0$, then α is projectively flat.*
Proof: If $(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0$, then

$$a_{ml}\alpha^2 G^m_\alpha = y_m y_l G^m_\alpha, \quad (6)$$

Contracting (6) with α^{lj}, we get

$$\alpha^2 G^j_\alpha = y_m y^j G^m_\alpha,$$

Let $\lambda(x,y) = \frac{y_m G^m_\alpha}{\alpha^2}$, then

$$G^j_\alpha = \lambda y^j.$$

Thus, α is projectively flat.

By (5) equation, we can prove the following theorem:

Theorem 3.3. *The (α,β)-metric $F=\alpha+\beta+\beta\,tan^{-1}(\beta/\alpha)$ is locally projectively flat iff*
(i) β is parallel with respect to α,
(ii) α is locally projectively flat, i.e., of constant curvature.
Proof: Suppose that F is locally projectively flat. First, we rewrite (5) as a polynomial in y^i and α. This gives

$((1+2b^2)\alpha^2-\beta^2)(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha((\alpha^2+\beta^2)(1+tan^{-1}(\beta/\alpha))+\alpha\beta)((1+2b^2)\alpha^2-\beta^2)s_{l0}+$

$$\alpha\{-2((\alpha^2+\beta^2)(1+tan^{-1}(\beta/\alpha))+\beta\alpha)s_0+\alpha r_{00}\}(b_l\alpha^2-\beta y_l)=0. \quad (7)$$

The coefficients of α *must be zero (note that* α^{even} *is a polynomial in* y^i *). We get*
$(1+2b^2)\alpha^2(a_{ml}\alpha^2-y_my_l)G^m_\alpha=-\alpha(\alpha^2+\beta^2)(1+tan^{-1}(\beta/\alpha))((1+2b^2)\alpha^2-\beta^2)s_{l0}$

$$+2\alpha s_0(\alpha^2+\beta^2)(1+tan^{-1}(\beta/\alpha))(b_l\alpha^2-\beta y_l)-\alpha^2 r_{00}(b_l\alpha^2-\beta y_l), \quad (8)$$

and

$$-\beta^2(a_{ml}\alpha^2-y_my_l)G^m_\alpha=-\alpha^2\beta((1+2b^2)\alpha^2-\beta^2)s_{l0}+2\alpha^2\beta s_0(b_l\alpha^2-\beta y_l). \quad (9)$$

Contracting (8) and (9) with b^l *yields*
$(1+2b^2)\alpha^2(b_m\alpha^2-y_m\beta)G^m_\alpha=-\alpha(\alpha^2+\beta^2)(1+tan^{-1}(\beta/\alpha))((1+2b^2)\alpha^2-\beta^2)s_0$

$$+2\alpha s_0(\alpha^2+\beta^2)(1+tan^{-1}(\beta/\alpha))(b^2\alpha^2-\beta^2)-\alpha^2 r_{00}(b^2\alpha^2-\beta^2), \quad (10)$$

and

$$-\beta^2(b_m\alpha^2-y_m\beta)G^m_\alpha=-\alpha^2\beta((1+2b^2)\alpha^2-\beta^2)s_0+2\alpha^2\beta s_0(b^2\alpha^2-\beta^2). \quad (11)$$

Rewrite (11) equation as

$$-\beta(b_m\alpha^2-y_m\beta)G^m_\alpha=-\alpha^4(1+s^2)s_0. \quad (12)$$

Note that β *is not divisible by* α^4 *and* $\alpha^4(1+s^2)$ *is not divisible by* β*. Thus* s_0 *is divisible by* β *and* $(b_m\alpha^2-y_m\beta)G^m_\alpha$ *is divisible by* α^4*. Therefore, there are scalar functions* $\tau=\tau(x)$ *and* $\chi=\chi(x)$ *such that*

$$(b_m\alpha^2-y_m\beta)G^m_\alpha=\chi\alpha^4, \quad (13)$$

$$s_0=\tau\beta. \quad (14)$$

Then (12) equation becomes

$$-\beta\chi\alpha^4=-\alpha^4(1+s^2)\tau\beta. \quad (15)$$

Since β *is not divisible by* $\alpha^4(1+s^2)$*. Thus* $\chi=\tau=0$*. So*

$$s_0=0, \quad (16)$$

and $(b_m\alpha^2 - y_m\beta)G^m_\alpha = 0$. *Now from (10), we have*

$$r_{00} = 0. \tag{17}$$

The substituting (16) and (17) into (8) and (9), we get

$$\begin{aligned}(1+2b^2)\alpha^2(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha \\ + \alpha(\alpha^2+\beta^2)(1+tan^{-1}(\beta/\alpha))((1+2b^2)\alpha^2-\beta^2)s_{l0} = 0,\end{aligned} \tag{18}$$

and

$$-\beta^2(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^2\beta((1+2b^2)\alpha^2-\beta^2)s_{l0} = 0. \tag{19}$$

Note that

$$\begin{vmatrix}(1+2b^2)\alpha^2 & \alpha(\alpha^2+\beta^2)(1+tan^{-1}(\beta/\alpha))((1+2b^2)\alpha^2-\beta^2) \\ -\beta^2 & \alpha^2\beta((1+2b^2)\alpha^2-\beta^2)\end{vmatrix} \neq 0.$$

Thus

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0, \tag{20}$$

$$s_{l0} = 0, \tag{21}$$

Then by Lemma 2.1 and Lemma 3.2, α *is projectively flat. And By (16) and (21) equation, we have* $b_{i|j} = 0$, *i.e.* β *is parallel with respect to* α.
On the converse, if β *is parallel with respect to* α *and* α *is locally projectively flat, then F is locally projectively flat.*

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