International Journal of Pure and Applied Mathematical Sciences.
ISSN 0972-9828 Volume 18, Number 2 (2025), pp. 95-102

https://www.ripublication.com

"Solution of Some Non Linear Differential Equations by Sadik Integral Transform & Adomian Method"

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Abstract

Integral Transform is useful technique of solving ordinary and partial differential equations, but for non linear differential equation have to combine Integral Transform with Adomian Polynomial. In this paper we will derive solution for different non linear differential equations by new integral transform that is Sadik transform combine with Adomian method, also we have presented application of this method.

Keywords: Sadik Transform, Adomian Polynomial and non-linear Differential Equation.

Introduction

There are various different methods for solving differential equation, but very less methods to solve non linear differential equations. So many researches are going on at the topic solution of non linear differential equations by Integral Transform with the help of Adomian polynomial. Laplace transform is the most effective tool to solve some kinds of ordinary and partial differential equations. Actually an electric engineer Oliver Heaviside made Laplace transform popular by developing its operational calculus. Likewise many integral transforms have been proposed which are similar to the Laplace transform, and each new transform claimed its own superiority over the Laplace transform. Shaikh Sadikali presented solution of some linear differential and partial differential equations by Sadik Transform. In this paper we considered a new integral transform named the Sadik transform. It is similar to the Laplace transform but the Laplace transform, the Sumudu transform, Elzaki transform and all integral transforms with kernel of an exponential type are particular cases of the Sadik transform. Due to the very general and unified nature of the Sadik transform, we can transport a problem of differential equations into the known transformation technique which is available in the literature through the Sadik transform.

Preliminaries

Sadik transform:

If, piecewise continuous on the interval $0 \le t \le A$ for any $A>0$. And $|(t)| \le K$.when$t \ge M$, for any real constant a. and some positive constant K and M. Then **Sadik transform** is defined by

$$F(\upsilon^{\alpha},\beta)=S[f(t)]=\frac{1}{\upsilon^{\beta}}\int_0^{¥} e^{-\upsilon^{\alpha}t} f(t)dt, \qquad \text{for} \operatorname{Re}(\upsilon^{\alpha})>w^{a}$$

Sadik transform to different derivatives, if $F(\upsilon)$ is Sadik Transform of $f(t)$ then,

$$S\left[f^{'}(t)\right]=\upsilon^{\alpha}F(\upsilon)-\upsilon^{-\beta}f(0) \qquad \& \qquad S\left[f^{(n)}(t)\right]=\upsilon^{n\alpha}F(\upsilon)-\sum_{k=o}^{n-1}\upsilon^{k\alpha-\beta}f^{(n-1)-k}(0)"$$

Adomain decomposition method

The Adomian decomposition method introduces the solution by decomposing $u(t)$ to an infinite series $u(t)=\sum_{n=0}^{\infty}u_n(t)$ and the nonlinear term Nu by the infinite series $Nu=\sum_{n=0}^{\infty}A_n$ where An are the Adomian polynomials which are generated for each nonlinear and can be found by the formula

$$A_i=\frac{1}{i!}\left[\frac{d_i}{d\lambda^i}V\left(\sum_{j=0}^{\infty}\lambda^j u_j\right)\right]_{\lambda=0}, i=0,1,2,...$$

Main Result

Theorem 1: Consider a nonlinear differential equation of general form,

$$L\left[u(t)\right]+U\left[u(t)\right]+N\left[u(t)\right]=g(t) \tag{1}$$

with initial condition $u(0)=f(t)$ (2)

where L is first order differential operator, U is linear function of $u(t)$ and N is nonlinear term containing nonlinear differential operator or $u(t)$. Then show that solution by Sadik transform with ADM is infinite series

$$u(t)=\sum_{n=0}^{\infty}u_n(t)=u_0(t)+u_1(t)+u_2(t)+\cdots \qquad \text{Where}$$

$$u_0(t)=f(t)+\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[g(t)\right]\right] \quad \& \quad u_{n+1}(t)=-\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[U\left(u_n(t)\right)+A_n\right]\right]$$

Proof: Applying Sadik transform on the equation (1)

$$\upsilon^{\alpha}\mathbb{S}\left[u(t)\right]-\upsilon^{-\beta}u(0)+\mathbb{S}\left[U\left[u(t)\right]\right]+\mathbb{S}\left[Nu(t)\right]=\mathbb{S}\left[g(t)\right] \tag{3}$$

$$\mathbb{S}\left[u(t)\right]=\frac{1}{\upsilon^{\alpha+\beta}}f(t)+\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[g(t)\right]-\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[U\left(u(t)\right)+N\left(u(t)\right)\right] \tag{4}$$

if we apply inverse Sadik transform we get the solution $u(t)$ but for nonlinear part we require to use Adomian polynomial method for which solution $u(t)$ by infinite series

$$u(t)=\sum_{i=0}^{\infty}u_i(t) \tag{5}$$

$$N\left(u(t)\right)=\sum_{i=0}^{\infty}A_i \tag{6}$$

$$\mathbb{S}\left[\sum_{i=0}^{\infty}u_i(t)\right]=\frac{1}{\upsilon^{\alpha+\beta}}f(t)+\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[g(t)\right]-\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[U\sum_{i=0}^{\infty}u_i(t)+\sum_{i=0}^{\infty}A_i\right] \tag{7}$$

$$\sum_{i=0}^{\infty}u_i(t)=f(t)+\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[g(t)\right]\right]-\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[U\sum_{i=0}^{\infty}u_i(t)+\sum_{i=0}^{\infty}A_i\right]\right] \tag{8}$$

comparing both sides for i=0 in (8) we get

$$u_0(t)=f(t)+\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[g(t)\right]\right] \tag{9}$$

then again (8) is given by

$$\sum_{i=1}^{\infty}u_i(t)=-\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[U\sum_{i=0}^{\infty}u_i(t)+\sum_{i=0}^{\infty}A_i\right]\right] \tag{10}$$

$$u_{n+1}(t)=-\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[U\left(u_n(t)\right)+A_n\right]\right] \tag{11}$$

Theorem 2: Consider a nonlinear differential equation (1)

$L\left[u(t)\right]+U\left[u(t)\right]+N\left[u(t)\right]=g(t)$ with initial conditions

$$u(0)=f_1(t),\ u'(0)=f_2(t).....u^{(n-1)}(0)=f_2(t) \tag{12}$$

where L is n^{th} order differential operator, U is reminder of differential operator who's order is less than n or linear function of $u(t)$ and N is nonlinear term containing nonlinear differential operator of $u(t)$ then show that by Sadik transform with ADM solution is infinite converges series

$u(t)=\sum_{n=0}^{\infty}u_n(t)=u_0(t)+u_1(t)+u_2(t)+\cdots$ where

$$u_0(t) = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{n\alpha}}\left(\sum_{k=0}^{n-1}\upsilon^{k\alpha-\beta}u^{n-k-1}(0) + \mathbb{S}\left[g(t)\right]\right)\right] \quad \&$$

$$u_{n+1}(t) = -\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{n\alpha}}\mathbb{S}\left[U\left(u_n(t)\right) + A_n\right]\right]$$

Proof: Applying Sadik transform to (1)

$$\upsilon^{n\alpha}\mathbb{S}\left(\upsilon^{\alpha},\beta\right) - \sum_{k=0}^{n-1}\upsilon^{k\alpha-\beta}u^{(n-1)-k}(0) + \mathbb{S}\left[U\left(u(t)\right)\right] + \mathbb{S}\left[N\left(u(t)\right)\right] = \mathbb{S}\left[g(t)\right]$$

$$\mathbb{S}\left[u(t)\right] = \frac{1}{\upsilon^{n\alpha}}\left(\sum_{k=0}^{n-1}\upsilon^{k\alpha-\beta}u^{(n-1)-k}(0) + S\left[g(t)\right]\right) - \frac{1}{\upsilon^{n\alpha}}\mathbb{S}\left[U\left(u(t)\right) + N\left(u(t)\right)\right] \quad (13)$$

where $\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{(n+1)\alpha+\beta}}\right] = \frac{t^n}{n!}$

if we apply inverse Sadik transform we get the solution $u(t)$ but for nonlinear part we require to use Adomian polynomial method for which we represent solution $u(t)$ by infinite series $u(t) = \sum_{i=0}^{\infty}u_i(t) \quad \& \quad N\left(u(t)\right) = \sum_{i=0}^{\infty}A_i$

where A_i are Adomian polynomial, substituting (5) and (6) and initial condition in (13)

$$\mathbb{S}\left[\sum_{i=0}^{\infty}u_i(t)\right] = \frac{1}{\upsilon^{n\alpha}}\left(\sum_{k=0}^{n-1}\upsilon^{k\alpha-\beta}u^{n-k-1}(0) + \mathbb{S}\left[g(t)\right]\right) - \frac{1}{\upsilon^{n\alpha}}\mathbb{S}\left[U\sum_{i=0}^{\infty}u_i(t) + \sum_{i=0}^{\infty}A_i\right]$$

$$\sum_{i=0}^{\infty}u_i(t) = F(t) - \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{n\alpha}}\mathbb{S}\left[U\sum_{i=0}^{\infty}u_i(t) + \sum_{i=0}^{\infty}A_i\right]\right] \quad (14)$$

where

$$F(t) = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{n\alpha}}\left(\sum_{k=0}^{n-1}\upsilon^{k\alpha-\beta}u^{(n-1)-k}(0) + \mathbb{S}\left[g(t)\right]\right)\right] \quad (15)$$

comparing both sides for i=0 of (14) we get,

$$u_0(t) = F(t) = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{n\alpha}}\left(\sum_{k=0}^{n-1}\upsilon^{k\alpha-\beta}u^{(n-1)-k}(0) + \mathbb{S}\left[g(t)\right]\right)\right] \quad (16)$$

then again (14) is given by

$$\sum_{i=1}^{\infty}u_i(t) = -\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{n\alpha}}\mathbb{S}\left[U\sum_{i=0}^{\infty}u_i(t) + \sum_{i=0}^{\infty}A_i\right]\right] \quad (17)$$

comparing both sides for i= (n+1) we get

$$u_{n+1}(t) = -\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{n\alpha}}\mathbb{S}\left[U\left(u_n(t)\right)+A_n\right]\right] \tag{18}$$

Application of the method

By applying Sadik Transform based on Adomian decomposition we solve non–linear differential equations with initial (value problem) conditions.

Example 1: Consider a nonlinear differential equation

$$\frac{d}{dt}u(t)=e^{u(t)} \quad \text{with the initial condition} \quad u(0)=0 \tag{19}$$

to find solution of (19) by Sadik transform we apply (9) and (11) here

$$\sum_{i=0}^{\infty} A_i = e^{u} \ ,$$

$u(0)=u_0(t)=0$ and hence $A_0 = e^0 = 1$ also $u_{n+1}(t) = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[A_k\right]\right]$

now $u_1(t) = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[A_0\right]\right] = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[1\right]\right] = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{2\alpha+\beta}}\right] = t$

using Adomian polynomial given by (3) where A_i are Adomian polynomial

$$A_1 = \frac{1}{1!}\left[\frac{d}{d\lambda}N\left(u_0+\lambda u_1\right)\right]_{\lambda=0} = \left[\frac{d}{d\lambda}e^{\left(u_0+\lambda u_1\right)}\right]_{\lambda=0} = e^{u_0}\ u_1 = t$$

$$u_2(t) = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[A_1\right]\right] = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t\right]\right] = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\cdot\frac{1}{\upsilon^{2\alpha+\beta}}\right] = \frac{t^2}{2}$$

$$A_2 = \frac{1}{2!}\left[\frac{d^2}{d\lambda^2}N\left(u_0+\lambda u_1+\lambda^2 u_2\right)\right]_{\lambda=0} = \frac{1}{2!}\left[\frac{d^2}{d\lambda^2}e^{u_0+\lambda u_1+\lambda^2 u_2}\right]_{\lambda=0}$$

$$A_2 = \frac{1}{2}e^{u_0}\left[\left(u_1\right)^2+\left(2u_2\right)\right] = \frac{1}{2}\left[t^2+2\frac{t^2}{2}\right] = t^2$$

$$u_3(t) = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[A_2\right]\right] = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t^2\right]\right] = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\cdot\frac{2!}{\upsilon^{3\alpha+\beta}}\right] = \frac{2t^3}{3!} = \frac{t^3}{3}$$

$$A_3 = \frac{1}{3!}\left[\frac{d^3}{d\lambda^3}N\left(u_0+\lambda u_1+\lambda^2 u_2+\lambda^3 u_3\right)\right]_{\lambda=0} = \frac{1}{3!}\left[\frac{d^3}{d\lambda^3}e^{u_0+\lambda u_1+\lambda^2 u_2+\lambda^3 u_3}\right]_{\lambda=0}$$

$$A_3 = \frac{1}{3!}\left[e^{u_0}\left(\left(u_1\right)^3+3\left(u_1\right)\left(2u_2\right)+6u_3\right)\right] = t^3$$

$$\therefore u_4(t) = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[A_3\right]\right] = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\mathbb{S}\left[t^3\right]\right] = \mathbb{S}^{-1}\left[\frac{1}{\upsilon^{\alpha}}\cdot\frac{3!}{\upsilon^{4\alpha+\beta}}\right] = \frac{3!}{4!}t^4 = \frac{t^4}{4}$$

similarly we can find $u_5(t)=\frac{t^5}{5}, u_6(t)=\frac{t^6}{6}\cdots\cdot$

then solution is convegece series $u(t)=\sum_{n=0}^{\infty}u_n(t)=u_0(t)+u_1(t)+u_2(t)+\cdots$

$$u(t)=t+\frac{t^2}{2}+\frac{t^3}{3}+\frac{t^4}{4}+\frac{t^5}{5}+\ldots\ldots$$

$$\therefore u(t)=-\log(1-t) \tag{20}$$

Example 2: Consider a nonlinear differential equation

$u''+(u')^2+(u)^2=1-\sin t$ with the conditions $u(0)=0$ and $u'(0)=1$ (21)

to find solution of (21) by Sadik transform we apply (16) and (18) here

$$u_0(t)=u(0)+t.u'(0)+\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{2\alpha}}\mathbb{S}[1-\sin t]\right] \tag{22}$$

$$u_{n+1}(t)=-\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{2\alpha}}\mathbb{S}\left[\left((u_t)^2\right)_n+\left(u^2\right)_n\right]\right] \tag{23}$$

$$\therefore u_0(t)=t+\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{2\alpha}}\left(\frac{1}{\upsilon^{\alpha+\beta}}-\frac{\upsilon^{-\beta}}{\upsilon^{2\alpha}+1}\right)\right]=t+S^{-1}\left(\frac{\upsilon^{-\beta}}{\upsilon^{3\alpha}}\right)-S^{-1}\left(\frac{\upsilon^{-\beta}}{\upsilon^{2\alpha}\left(\upsilon^{2\alpha}+1\right)}\right)$$

$$\therefore u_0(t)=t+\frac{t^2}{2!}-S^{-1}\left(\frac{\upsilon^{-\beta}\left(1+\upsilon^{2\alpha}-\upsilon^{2\alpha}\right)}{\upsilon^{2\alpha}\left(\upsilon^{2\alpha}+1\right)}\right)=\frac{t^2}{2!}+\sin t$$

$$\therefore (u_0)_t=t+\cos t$$

we have two nonlinear terms

$$u_{n+1}(t)=-\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{2\alpha}}\mathbb{S}\left[(A)_n+(B)_n\right]\right]$$

where $(A)_n=\left((u_t)^2\right)_n$ and $(B)_n=\left(u^2\right)_n$

$\therefore A_0=(t+\cos t)^2$ and $B_0=\left(\frac{t^2}{2!}+\sin t\right)^2$

$$u_1(t)=-\mathbb{S}^{-1}\left[\frac{1}{\upsilon^{2\alpha}}\mathbb{S}\left[(t+\cos t)^2+\left(\frac{t^2}{2!}+\sin t\right)^2\right]\right]$$

$$\therefore u_1(t)=-\frac{t^2}{2!}-\frac{2t^4}{4!}+\frac{t^6}{5!}+\cdots\cdot$$

similarly we can find $u_2(t),\ u_3(t)\cdots$

then solution is convegece series $u(t)=\sum_{n=0}^{\infty}u_n(t)=u_0(t)+u_1(t)+u_2(t)+\cdots$

$$u(t)=\frac{t^2}{2!}+\sin t-\frac{t^2}{2!}-\frac{2t^4}{4!}+\frac{t^6}{5!}+\cdots=\sin t$$

since $u(t)=\sin t$ satisfy the given differential and its initial conditions which represent the exact solution of given differential equation.

Conclusion

Since Solving Nonlinear differential equations is very typical, there is no single method exist which is useful to solve all types of Nonlinear differential equations, also there does not exist any transform through which we can solve nonlinear differential equations, all the transform who claim to solve any nonlinear differential equation are dependent on some supplementary methods like Adomian polynomial method or Homotopy perturbation method etc. In this chapter we have showed solution of some types of nonlinear Differential equation by Sadik Transform combined with ADM, also successfully we have solved its applications also; we have demonstrated solution of general nonlinear differential equation by Sadik Transform with ADM

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