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Fractional Integral Representation of Riemann Zeta Function

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Abstract

This paper presents the Riemann zeta function as a fractional integral by extending the Mellin transform representation to any positive real numbers and then to any complex numbers having their real part greater than zero. The Riemann-Liouville definition of the fractional integral is used to extend the representation.

Keywords: Fractional integral, Riemann zeta function, Mellin transform.

1. INTRODUCTION

Fractional calculus is the field of mathematical analysis that studies how to generalize the operations of differentiation and integration from the integer order to arbitrary order $\alpha \in \mathbb{R}$ or $\mathbb{C}$. This generalization provides a useful tool for handling real-world problems that can't be adequately described by the classical integer-order models. The Riemann-Liouville and Caputo formulations are among the most widely used definitions of fractional derivatives and integrals [1, 2]. In this paper, we used the Riemann-Liouville formulation to represent the Riemann zeta function. In recent decades, fractional calculus has gained significant attention across various scientific and engineering disciplines. The growing availability of computational methods for solving fractional differential equations with the aid of numerical techniques and transform methods such as the Laplace, Fourier, and Mellin transforms has made fractional calculus an indispensable tool in both theoretical and applied research [3, 4].

The Riemann zeta function occupies a central place in analytic number theory and complex analysis, bridging the gap between arithmetic and analysis through the study of prime numbers and infinite series. It was first introduced by Leonhard Euler in the $18th$ century in connection with the sum of reciprocal powers of natural numbers, and later extended by Bernhard Riemann in his seminal 1859 paper "On the Number of Primes Less Than a Given Magnitude" [5- 8]. For a complex variable $s = \sigma + it$ with $\Re(s) > 1$, where $\Re(s)$ denotes the real part of s, the Riemann zeta function is defined by the convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Several authors have published works on different representations for Riemann zeta function which is defined as an analytic continuation of the generalized harmonic series $\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$, $s > 1$. One of the prominent integral representations of the Riemann zeta function is

$$\Gamma(s)\,\zeta(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx, \;\; \Re(s) > 1$$

where Γ denotes the gamma function. This is one of the various representations of the zeta function as a Mellin transform [9-11] and it is valid when the real part of s is greater than one. Riemann showed that

$$2\,\sin(\pi s)\,\Gamma(s)\,\zeta(s) = i \int_C \frac{(-x)^{s-1}}{e^x - 1} dx$$

for all s, where the contour C starts and ends at $+\infty$ and it circles the origin once. We can find representations in the literature that relate to the prime-counting function $\pi(x)$ [12] in the form

$$log\,\zeta(s) = s \int_0^{\infty} \frac{\pi(x)}{x(x^s - 1)} dx$$

for all $\Re(s) > 0$. Here $\pi(x)$ counts the number of primes less than a given real number x. Another representation of the zeta function is given by

$$log\,\zeta(s) = s \int_0^{\infty} J(x)\, x^{-s-1} dx$$

where $J(x)$ is the Riemann prime powers p^n counting function. The Riemann zeta function can be expressed as a divergent Mellin transform as

$$2\pi^{-s/2}\Gamma(s/2)\,\zeta(s) = \int_0^{\infty} \theta(ix)\, x^{s/2-1} dx$$

in terms of Jacobi's theta function

$$\theta(ix) = \sum_{-\infty}^{\infty} e^{in^2 \pi x}$$

This integral does not converge for any value of s, and a regularization of this integral gives the following representation

$$\pi^{-s/2}\Gamma(s/2)\,\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \frac{1}{2}\int_0^1 (\theta(ix) - x^{-1/2})\,x^{s/2-1}dx + \frac{1}{2}\int_1^\infty (\theta(ix) - 1)\,x^{s/2-1}dx$$

2. MULTIPLE INTEGRAL REPRESENTATION OF RIEMANN ZETA FUNCTION

Consider the series

$$U(x) = \sum_{n=1}^{\infty} e^{-nx}, \quad \text{for } x > 0$$

Clearly $U(x)$ converges absolutely for all $x > 0$ and

$$U(x) = e^{-x} + e^{-2x} + e^{-3x} + \cdots = \frac{1}{e^x - 1}\cdot$$

Integrating $U(x)$ from x to ∞, we get

$$\int_x^\infty U(t)dt = \int_x^\infty \frac{1}{e^t - 1}dt$$

Since the series $U(x)$ is absolutely convergent, we can interchange summation and integration, so that

$$\sum_{n=1}^{\infty}\int_x^\infty e^{-nt}dt = \int_x^\infty \frac{1}{e^t - 1}dt$$

That is,

$$\sum_{n=1}^{\infty}\frac{e^{-nx}}{n} = \int_x^\infty \frac{1}{e^t - 1}dt \tag{1}$$

Applying the limit as x tends to zero, we get

$$\zeta(1) = \sum_{n=1}^{\infty}\frac{1}{n} = \lim_{x\to 0}\int_x^\infty \frac{1}{e^t - 1}dt \tag{2}$$

Integrating both sides of equation (1) from x to infinity, we get

$$\sum_{n=1}^{\infty}\frac{e^{-nx}}{n^2} = \int_x^\infty\int_x^\infty \frac{1}{e^t - 1}dt\,dt \tag{3}$$

Applying limit as x tends to zero, we get

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \lim_{x\to 0}\int_x^\infty\int_x^\infty \frac{1}{e^t - 1}dt\,dt \tag{4}$$

Suppose we perform $s-2$ number of successive integration of both sides of (3) from x to infinity and interchanging summation and integration on the left side, we get

$$\sum_{n=1}^{\infty}\frac{e^{-nx}}{n^s}=\int_x^{\infty}\cdots\int_x^{\infty}\frac{1}{e^t-1}dt\cdots dt \tag{5}$$

Here we use the notation

$$\int_x^{\infty}\cdots\int_x^{\infty}\frac{1}{e^t-1}dt\cdots dt=\int_x^{\infty}\frac{1}{e^t-1}dt^s$$

Then, equation (5) takes the form

$$\sum_{n=1}^{\infty}\frac{e^{-nx}}{n^s}=\int_x^{\infty}\frac{1}{e^t-1}dt^s \tag{6}$$

Applying limit as x tends to zero, we get

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\lim_{x\to 0}\int_x^{\infty}\frac{1}{e^t-1}dt^s \tag{7}$$

A simple application of the principle of mathematical induction shows that the above result is true for any positive integer. Thus, we have the following theorem:

Theorem 2.1. *Let s be a positive integer, then*

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\lim_{x\to 0}\int_x^{\infty}\frac{1}{e^t-1}dt^s$$

Next, we generalize the theorem (2.1) for any positive real number s, and then to complex numbers in the half plane $\Re(s)>0$.

3. FRACTIONAL INTEGRAL REPRESENTATION OF RIEMANN ZETA FUNCTION

In this section, we extend the representation of the Riemann zeta function to any real number greater than zero using the Riemann-Liouville fractional integral and then to any complex number whose real part is greater than zero. Here, we use the analytic continuation of a meromorphic function with the help of Riemann-Liouville fractional derivative.

Lemma 3.1. *Let a be a constant, s be a positive integer, and $f(x)$ be an integrable function in $[a,x]$, then the s-fold integral*

$$\int_a^x f(x)dx^s=\frac{1}{\Gamma(s)}\int_a^x(x-t)^{s-1}f(t)dt$$

Proof:
We can prove this result using the principle of mathematical induction on $s \in \mathbb{N}$.

For a positive integer n, let us denote the n-fold integral $\int_a^x \int_a^x \cdots \int_a^x f(t)dt^n$ by $(I^n f)(x)$. That is, $(I^n f)(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt$

For $n = 1$,

$$(I^1 f)(x) = \frac{1}{\Gamma(1)} \int_a^x (x-t)^{1-1} f(t)dt = \int_a^x f(t)dt$$

So the case $n = 1$ holds.

Assume that the formula holds for some $n > 1$. That is,

$$\begin{aligned}
(I^n f)(x) &= \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt \\
&= \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt \\
\text{Consider } (I^{n+1} f)(x) &= I(I^n f)(x) \\
&= \int_a^x (I^n f)(t)dt \\
&= \int_a^x \left(\frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau)d\tau\right)dt \\
&= \frac{1}{(n-1)!} \int_a^x \int_a^t (t-\tau)^{n-1} f(\tau)\, d\tau\, dt
\end{aligned}$$

Since f is continuous, we can apply Fubini's theorem to swap the order of integration. Changing the order of integration from $d\tau\, dt$ to $dt\, d\tau$, we have

$$\begin{aligned}
(I^{n+1} f)(x) &= \frac{1}{(n-1)!} \int_a^x \int_\tau^x (t-\tau)^{n-1} f(\tau)\, dt\, d\tau \\
&= \frac{1}{(n-1)!} \int_a^x f(\tau) \left[\int_\tau^x (t-\tau)^{n-1}\, dt\right] d\tau \\
&= \frac{1}{(n-1)!} \int_a^x f(\tau) \frac{(x-\tau)^n}{n}\, d\tau \\
&= \frac{1}{n!} \int_a^x (x-\tau)^n f(\tau)\, d\tau \\
&= \frac{1}{\Gamma(n+1)} \int_a^x (x-\tau)^{(n+1)-1} f(\tau)\, d\tau
\end{aligned}$$

Thus, by induction, the formula holds for every positive integer n. □

We know that the poles of $\frac{1}{\Gamma(s)}$ are zero and negative integers. Hence, if $s > 0$, we can extend the result given in Lemma 3.1 to any real number $s > 0$ using the Riemann-Liouville definition of fractional integration, and hence we have the following theorem.

Theorem 3.1. *Let* $s > 0$ *be any real number, then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \lim_{x \to 0} \int_x^{\infty} \frac{1}{e^t - 1} dt^s = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

Proof:

If $s > 0$ and f is integrable on $[a, x]$, by the Riemann-Liouville version of fractional integration,

$$(I^s f)(x) = \frac{1}{\Gamma(s)} \int_a^x (x - t)^{s-1} f(t) dt$$

Note that

$$\begin{aligned} \int_x^{\infty} \frac{1}{e^t - 1} dt^s &= \lim_{b \to \infty} \int_b^x \frac{(-1)^s}{e^t - 1} dt^s \\ &= \lim_{b \to \infty} \frac{1}{\Gamma(s)} \int_b^x \frac{(-1)^s (x - t)^{s-1}}{e^t - 1} dt \\ &= \frac{1}{\Gamma(s)} \int_x^{\infty} \frac{(-1)^{s+1} (x - t)^{s-1}}{e^t - 1} dt \\ &= \frac{1}{\Gamma(s)} \int_x^{\infty} \frac{(t - x)^{s-1}}{e^t - 1} dt \end{aligned}$$

Therefore,

$$\lim_{x \to 0} \int_x^{\infty} \frac{1}{e^t - 1} dt^s = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

Thus, if s is any real number greater than zero, we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \lim_{x \to 0} \int_x^{\infty} \frac{1}{e^t - 1} dt^s = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt \qquad \square$$

Lemma 3.2. *Let f be an integrable function on* $[a, x]$ *and s any complex number whose* $\Re(s) > 0$, *then the Riemann-Liouville fractional integral of the complex order s is*

$$(I^s f)(x) = \frac{1}{\Gamma(s)} \int_a^x (x - t)^{s-1} f(t) dt$$

Proof:

The gamma function $\Gamma(s)$ is analytic except at the simple poles $\{0, -1, -2, \cdots\}$. If $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, first of all we establish the semigroup property (beta convolution). That is,

$$I^{\alpha}(I^{\beta} f) = (I^{\alpha+\beta} f) \text{ for } \Re(\alpha) > 0, \Re(\beta) > 0$$

$$I^{\alpha}(I^{\beta} f)(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha-1}\Big(\int_a^t (t-\tau)^{\beta-1} f(\tau) d\tau\Big) dt$$

Since f is locally integrable and the kernel functions are positive, we can change the order of integration using Fubini's theorem. Thus,

$$\begin{aligned} I^{\alpha}(I^{\beta} f)(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha-1}\Big(\int_a^t (t-\tau)^{\beta-1} f(\tau) d\tau\Big) dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\tau) \int_\tau^x \Big((x-t)^{\alpha-1}(t-\tau)^{\beta-1} dt\Big) d\tau \end{aligned}$$

Use the substitution $u = \dfrac{t-\tau}{x-\tau}$.

$u = \dfrac{t-\tau}{x-\tau} \implies 1-u = \dfrac{x-t}{x-\tau}$ and $(x-\tau)du = dt$.

Now the inner integral,

$$\begin{aligned} \int_\tau^x (x-t)^{\alpha-1}(t-\tau)^{\beta-1} dt &= \int_0^1 (1-u)^{\alpha-1}(x-\tau)^{\alpha-1} u^{\beta-1}(x-\tau)^{\beta-1}(x-\tau) du \\ &= (x-\tau)^{\alpha+\beta-1} \int_0^1 (1-u)^{\alpha-1} u^{\beta-1} du \\ &= (x-\tau)^{\alpha+\beta-1} B(\alpha+\beta) \\ &= (x-\tau)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \end{aligned}$$

So, we get

$$\begin{aligned} I^{\alpha}(I^{\beta} f)(x) &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-\tau)^{\alpha+\beta-1} f(\tau)\, d\tau \\ &= (I^{\alpha+\beta} f)(x) \end{aligned}$$

Now for $\Re(s) > 0$ the mapping $s \to I^s f$ can be analytically continued in s to a meromorphic vector valued-function on $\mathbb{C}$ whose only singularities come from the poles of $\Gamma(s)$. That is, for each fixed $x > a$, the quantity

$$\Gamma(s)(I^s f)(x) = \int_a^x (x-t)^{s-1} f(t) dt$$

is an entire function of s because the integrand depends holomorphically on s and the integral converges for $\Re(s) > 0$. Hence, $I^s f$ is meromorphic in s with possible simple poles where $\Gamma(s) = 0$.

Because the semigroup property holds on the half plane where both sides are defined, analytic continuation extends the semigroup identity to larger domains in s except for singularities. So we may use operator relations like $I^\alpha I^\beta = I^{\alpha+\beta}$ for complex α,β after meromorphic continuation, avoiding poles.

Now we can define the Riemann-Liouville fractional derivative of order $s \in \mathbb{C}$ by

$$(D^s f)(x) = \frac{d^n}{dx^n}(I^{n-s} f)(x) \tag{8}$$

where $n = \lceil \Re(s) \rceil$, the smallest integer $\geq \Re(s)$.

In the right hand side of equation (8), the fractional integral I^{n-s} makes sense because $\Re(n-s) > 0$. Then apply the ordinary n^{th} derivative, where n is an integer. When $s = n$, equation (8) is the natural generalization of integer order differentiation, and hence we get $D^n f = \frac{d^n}{dx^n}$.

Theorem 3.2. *Let s be any complex number whose* $\Re(s) > 0$, *then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \lim_{x\to 0} \int_x^{\infty} \frac{1}{e^t - 1} dt^s = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

Proof:

If s is any complex number whose $\Re(s) > 0$, by the Lemma 3.2 we have

$$(I^s f)(x) = \frac{1}{\Gamma(s)} \int_a^x (x-t)^{s-1} f(t) dt$$

Combining this with the proof of Theorem 3.1, we have the result

$$\lim_{x\to 0} \int_x^{\infty} \frac{1}{e^t - 1} dt^s = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

Thus,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \lim_{x\to 0} \int_x^{\infty} \frac{1}{e^t - 1} dt^s = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

for any complex number s satisfying $\Re(s) > 0$. □

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