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Fixed Point Theorems in Bi Two Metric Spaces and Quasi Partial S_b Metric Spaces

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Abstract

The aim of this communication is to treat a study on some new class of fixed point theorems under self mappings in bi two metric and complete quasi partial S_b-metric spaces with some conditions, which generalize some of the existing results in the literature of fixed point theorems in that metric spaces. We have also given some examples to explain our points.

Key words: Fixed point theory, Self mapping, Bi two metric space, Complete quasi partial S_b-metric space, T-orbitally continuous, T-orbitally complete, Converging point.

1. Introduction

In 1922, the well-known concept of contraction principle was first time introduced by Banach [2]. This principle is one of the simplest and most useful methods for the construction of solutions of linear and non-linear equations. Fixed point theory is a beautiful mixture of mathematical analysis to explain some conditions in which mapps give excellent solutions. This theory not only plays an important role in existence problems in many branches of mathematical analysis but also helps inside and outside the mathematics

In this paper, we consider the concept of a two metric space as considered by Gahler [7] and prove some fixed point theorems in two metric spaces which generalize some of the existing results on fixed point theorems in metric spaces to two and bi-two metric spaces.

If X be any non empty set, equipped with two metric d, is said to be two metric space (X, ρ) if ρ satisfies the following properties

i. For the two distinct points $x, y \in X$, there exists a point $z \in X$such that $\rho(x, y, z) \neq 0$ and $\rho(x, y, z) = 0$ if atleast two of the three points are equal,

ii. $\rho(x,y,z)=\rho(x,z,y)=\rho(y,z,x)$, for all $x,y,z\in X$,
iii. $\rho(x,y,z)\leq\rho(x,y,a)+\rho(x,a,z)+\rho(a,y,z)$, for all $x,y,z,a\in X$.

Now, we starts with some definitions:

Definition 1.1 Let X be a two metric space and T be a mapping of X into itself. Then X is said to be T-orbitally complete if $\{TT^nx\}$ converges in X whenever $\{T^nx\}$ is Cauchy in X, $x\in X$.

Definition 1.2 A mapping T of a two metric space (X,ρ) into itself is said to be T-orbitally continuous on X if $\rho(T^nx,u.a)\to 0(i\to\infty)$ implies $\rho(TT^nx,Tu.a)\to 0(i\to\infty)$, for all $a\in X$.

Definition 1.3 Let (X,ρ) be a two metric space then it is said to be complete if every Cauchy sequence is convergent in X.

Definition 1.4 A two metric space (X,ρ) is said to be bounded if the set $\{\rho(x,y,z): x,y,z\in X\}$ is a bounded set of real number $\boldsymbol{R}$.

Definition 1.5 A sequence $\{x_n\}$ from a two metric space X is said to be converge to a point $x\in X$ if $\rho(x_n,x,a)\to 0$ when $n\to\infty$ for all $a\in X$. If $x_n\to x$ then we can say x is a limit of $\{x_n\}$.

Definition 1.5 Let (X,ρ) be a two metric space then a sequence $\{x_n\}$ from X is said to be Cauchy sequence if $\rho(x_m,x_n,a)\to 0$ when $m,n\to\infty$, for each $a\in X$.
There exists many generalizations in the literature of metric spaces. Later, many other researchers increase this idea and made more efforts in this particular area. For instance, Mlaiki [8, 9, 10] and Sedghi [11, 12] proposed the fixed point theory in S-Metric Spaces. Also, fixed point results in b-metric spaces were also studied by many researchers [1], [3], [4], [5] and [6].

Definition 1.4 A mapping $\rho_h: X^3\to[0,\infty)$, where X is a non-empty set, is said to be partial S_b-metric with coefficient $t\geq 1$ if whenever $\alpha,\beta,\gamma,\eta\in X$ having the following conditions: (i) $\alpha=\beta=\gamma$ if and only if $\rho_h(\alpha,\beta,\gamma)=\rho_h(\alpha,\alpha,\alpha)=\rho_h(\beta,\beta,\beta)=\rho_h(\gamma,\gamma,\gamma)$; (ii) $\rho_h(\alpha,\alpha,\alpha)\leq\rho_h(\alpha,\beta,\gamma)$; (iii) $\rho_h(\alpha,\alpha,\beta)\leq\rho_h(\beta,\beta,\alpha)$; (iv) $\rho_h(\alpha,\beta,\gamma)\leq t[\rho_h(\alpha,\alpha,\eta)+\rho_h(\beta,\beta,\eta)+\rho_h(\gamma,\gamma,\eta)]-\rho_h(\eta,\eta,\eta)$. Then, the pair (X,ρ_h) is called partial S_b-metric space with coefficient $t\geq 1$.

Definition 1.5 In a partial S_b-metric space (X,ρ_h): (i) A sequence $\{x_n\}$ is said to be convergent to x if $\lim\limits_{n\to\infty}\rho_h(x_n,x_n,x)=\rho_h(x,x,x)$; (ii) A sequence $\{x_n\}$ is said to be a Cauchy sequence in X if $\lim\limits_{n,m\to\infty}\rho_h(x_n,x_n,x_m)$ exists; (iii) A partial S_b-metric space

(X, ρ_h) is said to be complete if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that $\lim_{n\to\infty} \rho_h(x_n, x_n, x) = \rho_h(x, x, x) = \lim_{n,m\to\infty} \rho_h(x_n, x_n, x_m)$.

Definition 1.6 A mapping $\rho_h: X^3 \to [0, \infty)$, where X is a non-empty set, is said to be quasi partial S_b-metric space with coefficient $t \geq 1$ if whenever $\alpha, \beta, \gamma, \eta \in X$ having the following conditions: (i) $\alpha = \beta = \gamma$ if and only if $\rho_h(\alpha, \beta, \gamma) = \rho_h(\alpha, \alpha, \alpha) = \rho_h(\beta, \beta, \beta) = \rho_h(\gamma, \gamma, \gamma)$; (ii) $\rho_h(\alpha, \alpha, \alpha) \leq \rho_h(\alpha, \beta, \gamma)$; (iii) $\rho_h(\alpha, \beta, \gamma) \leq t[\rho_h(\alpha, \alpha, \eta) + \rho_h(\beta, \beta, \eta) + \rho_h(\gamma, \gamma, \eta)] - \rho_h(\eta, \eta, \eta)$. Then, the pair (X, ρ_h) is called quasi partial S_b-metric space with coefficient $t \geq 1$.

Example 1.7 If $\mathcal{X} = \{0,1,2,3\}$ and $\rho_h(p, q, r) = |q - p|^2 + |q - r|^2 + |r - p|^2 + p$. Now, we define $T: \mathcal{X} \to \mathcal{X}$ by $T0 = 0$, $T1 = 0$, $T2 = 1$ and $T3 = 2$. Ofcourse, ρ_h is a quasi partial S_b-metric space on $\mathcal{X}$. Apparently 0 is a unique fixed point for T.

2. Our Results

Theorem 2.1 Let (X, ρ) be a complete two metric space and let T_1 and T_2 be two self mappings of X satisfying the condition $\rho(T_1x, T_2y, a) \leq a_1 \max\{\rho(y, T_1x, a) + \rho(y, T_2y, a)\} + a_2 \max\{\rho(x, T_2y, a) + \rho(x, y, a), \rho(x, T_2y, a)\} + a_3\rho(x, T_1x, a)$.For all $x, y, a \in X$, $a_1 + a_2 + a_3 < 1$ and $a_1 + 2a_2 < 1$, then T_1 and T_2 have a common fixed point.

Proof: Let (X, ρ) be a complete two metric space and let T_1 and T_2 be two self mappings of X satisfying the condition $(T_1x, T_2y, a) \leq a_1 \max\{\rho(y, T_1x, a) + \rho(y, T_2y, a)\} + a_2 \max\{\rho(x, T_2y, a) + \rho(x, y, a), \rho(x, T_2y, a)\} + a_3\rho(x, T_1x, a)$. Let x_0 be an arbitrary point of X and we define the sequence $\{x_n\}$ of points of X such that $T_1x_{2n} = x_{2n+1}$, $T_2x_{2n+1} = x_{2n+2}$, $n \geq 0$. We have $\rho(x_{2n+1}, x_{2n+2}, a) = \rho(T_1x_{2n}, T_2x_{2n+1}, a) \leq a_1 \max\{\rho(x_{2n+1}, T_1x_{2n}, a) + \rho(x_{2n+1}, T_2x_{2n+1}, a)\} + a_2 \max\{\rho(x_{2n}, T_2x_{2n+1}, a) + \rho(x_{2n}, x_{2n+1}, a), \rho(x_{2n+1}, T_2x_{2n+1}, a)\} + a_3\rho(x_{2n}, T_1x_{2n}, a) = a_1 \max\{\rho(x_{2n+1}, x_{2n+1}, a) + \rho(x_{2n+1}, x_{2n+2}, a)\} + a_2 \max\{\rho(x_{2n}, x_{2n+2}, a) + \rho(x_{2n}, x_{2n+1}, a), \rho(x_{2n+1}, T_2x_{2n+2}, a)\} + a_3\rho(x_{2n}, x_{2n+1}, a) = a_1\rho(x_{2n+1}, x_{2n+2}, a) + a_2\rho(x_{2n+1}, x_{2n+2}, a) + a_3\rho(x_{2n}, x_{2n+1}, a)$ $i.e.$ $\rho(x_{2n+1}, x_{2n+2}, a) \leq \frac{a_3}{1-a_1-a_2}\rho(x_{2n}, x_{2n+1}, a)$ $i.e.$ $\rho(x_{2n+1}, x_{2n+2}, a) \leq h\rho(x_{2n}, x_{2n+1}, a)$ where $h = \frac{a_3}{1-a_1-a_2}$ Again,$a_1 + a_2 + a_3 < 1$ $i.e.$ $a_3 < 1 - a_1 - a_2$ that means $h = \frac{a_3}{1-a_1-a_2} < 1$ Sothat $\rho(x_{2n+1}, x_{2n+2}, a) < h\rho(x_{2n}, x_{2n+1}, a)$. Obviously $\rho(x_{2n+1}, x_{2n+2}, a) < h^2\rho(x_{2n}, x_{2n+1}, a)$, and so on. Hence $< h^{2n+1}\rho(x_0, x_1, a) \to 0$ when $n \to \infty$, since $h < 1$. Thus the sequence $\{x_n\}$ is a Cauchy sequence from X. Now, since X is complete. So $z \in X$ such that $\log_{n\to\infty} x_n = z$. Now, we have to show that z is a common fixed point of T_1 and T_2. That means $T_1(z) = T_2(z) = z$. Now, we have $(T_1z, x_{2n+2}, a) = \rho(T_1z, T_2x_{2n+1}, a) \leq a_1 \max\{\rho(x_{2n+1}, T_1z, a) + \rho(x_{2n+1}, T_2x_{2n+1}, a)\} + a_2 \max\{\rho(z, T_2x_{2n+1}, a) +$

$\rho(z, x_{2n+1}, a), \rho(x_{2n+1}, T_2x_{2n+1}, a)\} + a_3\rho(z, T_1z, a)$. Taking limit as $n \to \infty$ both sides we obtain $(T_1z, z, a) \leq a_1\rho(z, T_1z, a) + a_3\rho(z, T_1z, a)$ $i.e. \rho(T_1z, z, a) \leq (a_1 + a_3)(z, T_1z, a)$ $i.e. T_1z = z$, since $(a_1 + a_3) < 1$. Now, when $\rho(x_{2n+1}, T_2z, a)$ we obtain the result $T_2z = z$. Hence $T_1z = T_2z = z$. Thus, z is a common fixed point of T_1 and T_2.

Theorem 2.2 Let T be a self mapping of bi two metric space (X, ρ, ρ_1) satisfying the following conditions (i) $\rho_1(x, y, a) < \rho(x, y, a)$, for all $x, y, a \in X$, (ii) X is T-orbitally complete with respect to ρ_1, (iii) T is orbitally continuous with respect to ρ_1, and (iv) There exists a real number $r \in (0,1)$ such that

$$\alpha_1\rho(x, Tx, a)\rho(Tx, Ty, a) + \alpha_2\rho(x, y, a)\rho(x, Tx, a) + \alpha_3\rho(x, y, a)\rho(y, Ty, a)$$
$$+\alpha_4[\rho(x, y, a)]^2 + \alpha_5\rho(x, Tx, a)\rho(y, Ty, a) -$$
$$\min\{\rho(x, y, a)\rho(x, Tx, a), \rho(x, Ty, a)\rho(y, Tx, a), \rho(x, Tx, a)\rho(y, Ty, a), \rho(y, Tx, a)\rho(y, Ty, a)\}$$
$$\leq \beta\rho(x, y, a)\rho(x, Tx, a)$$

where $\alpha, \beta \in \boldsymbol{R}$ such that $\sum_{i=1}^{5} \alpha_i > \beta$ and $\beta - \alpha_2 - \alpha_4 \geq 0$. Then for each $x \in X$ there exists a sequence $\{T^n x\}$ converges to a fixed point of T.

Proof: Let X be a bi two metric space equipped with two metrics ρ and ρ_1. Let $x_0 \in X$. Now, we define a sequence $\{x_n\}$, where $x_n = T^n x_0$. If for some $n, x_n = x_{n+1} = x_{n+2}$ then x_n is a fixed point of T. Let $x_n \neq x_{n+1} \neq x_{n+2}$ for every $n \geq 0$. Now, from the given condition, taking $x = x_{n-1}, y = x_n$ we obtain $\alpha_1\rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a) + \alpha_2\rho(x_{n-1}, x_n, a)\rho(x_{n-1}, x_n, a) + \alpha_3\rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a) + \alpha_4[\rho(x_n, x_{n+1}, a)]^2 + \alpha_5\rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a) - \min\{\rho(x_{n-1}, x_n, a)\rho(x_{n-1}, x_n, a),$ $\rho(x_{n-1}, x_{n+1}, a)\rho(x_n, x_n, a), \rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a), \rho(x_n, x_n, a)\rho\big(x_n, x_{n+1,}x_{n+2}\big)\}$ $\leq \beta\rho(x_n, x_{n-1}, a)\rho(x_{n-1}, x_n, a)$ $i.e. (\alpha_1 + \alpha_3 + \alpha_5)\rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a) + (\alpha_2 + \alpha_4)[\rho(x_{n-1}, x_n, a)]^2 - [\rho(x_{n-1}, x_n, a)]^2 \leq \beta[\rho(x_{n-1}, x_n, a)]^2$ $i.e. (\alpha_1 + \alpha_3 + \alpha_5)\rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a) \leq (\beta - \alpha_2 - \alpha_4 + 1)[\rho(x_{n-1}, x_n, a)]^2$ $i.e. \rho(x_n, x_{n+1}, a) \leq \frac{(\beta - \alpha_2 - \alpha_4 + 1)}{(\alpha_1 + \alpha_3 + \alpha_5)}\rho(x_{n-1}, x_n, a)]$ $i.e. \rho(x_n, x_{n+1}, a) \leq r[\rho(x_{n-1}, x_n, a)]$ where $r = \frac{(\beta - \alpha_2 - \alpha_4 + 1)}{(\alpha_1 + \alpha_3 + \alpha_5)}$ now $\sum_{i=1}^{5} \alpha_i > \beta$, and $\beta - \alpha_2 - \alpha_4 \geq 0$. It implies that $r < 1$. That means $\rho(x_n, x_{n+1}, a) \leq r[\rho(x_n, x_{n+1}, a)]$ $i.e. \rho(x_n, x_{n+1}, a) \leq r[\rho(x_{n-1}, x_n, a)]$ $i.e. \rho(x_n, x_{n+1}, a) \leq r^2\rho(x_{n-2}, x_{n-1}, a) \leq \cdots \leq r^n(x_0, x_1 a)$.

Again, now $\rho(x_n, x_{n+m}, a) \leq \rho(x_n, x_{n+m}, x_{n+1}) + \rho(x_n, x_{n+1}, a) + \rho(x_{n+1}, x_{n+m}, a) = \rho(x_n, x_{n+1}, x_{n+m}) + \rho(x_n, x_{n+1}, a) + \rho(x_{n+1}, x_{n+m}, a) = \rho(x_n, x_{n+1}, x_{n+m}) + \rho(x_n, x_{n+1}, a) + \rho(x_{n+1}, x_{n+m}, x_{n+2}) + \rho(x_{n+1}, x_{n+2}, a) + \rho(x_{n+2}, x_{n+m}, a) = \cdots$and so on. Hence

$$\rho(x_n, x_{n+m}, a) \leq \rho(x_n, x_{n+1}, x_{n+m}) + \rho(x_n, x_{n+1}, a) + \rho(x_{n+1}, x_{n+m}, x_{n+2}) + \rho(x_{n+1}, x_{n+2}, a)$$
$$+ \cdots \rho(x_{n+m-2}, x_{n+m-1}, x_{n+m}) + \rho(x_{n+m-1}, x_{n+m}, a) = \sum_{k=n}^{n+m} r^k\rho(x_o, x_1, x_{n+m}) + \sum_{k=n}^{n+m} r^k$$

$\rho(x_0, x_1, a)$. Now, ρ is bounded. So $\rho(x, y, a) \leq N$ for some real number $N > 0$. $\rho(x_n, x_{n+m}, a) \leq 2N\sum_{k=n}^{n+m} r^k \leq 2N\frac{r^n}{1-r} \to 0$ when $n \to \infty$, since $r < 1$. This implies $\{x_n\}$ is a Cauchy sequence in (X, ρ_1). Now, X is T-orbitally complete with respect to

ρ_1, there exists $s \in X$ such that $\lim_{n\to\infty} x_n = s$. Again, T-orbitally continuous with respect to ρ_1 and this implies that $Ts = \lim_{n\to\infty} T^{n+1}x = s$. Hence s is a fixed point of T.

Theorem 2.3 Let T be a self mapping of bi two metric space (X, ρ, ρ_1) satisfying the conditions (i) $\rho_1(x, y, a) < \rho(x, y, a)$, for all $x, y, a \in X$, (ii) X is T-orbitally complete with respect to ρ_1, (iii) T is orbitally continuous with respect to ρ_1, and (iv) There exists a real number $r \in (0,1)$ such that: $\min\{\rho(x, Tx, a)\rho(Tx, Ty, a), \rho(x, y, a)\rho(x, Tx, a), \rho(x, y, a)\rho(y, Ty, a), [\rho(x, y, a)]^2, \rho(x, Tx, a)\rho(y, Ty, a)\} - \min\{\rho(x, y, a)\rho(x, Tx, a), \rho(x, Ty, a)\rho(y, Tx, a), \rho(x, Tx, a)\rho(y, Ty, a), \rho(y, Tx, a)\rho(y, Ty, a)\} \leq r\rho(x, y, a)\rho(x, Tx, a)$ for all distinct $x, y. a \in X$ there exists a sequence $\{T^n x\}$ which, converges to a fixed point of T.

Proof: Let X be a bi two metric space equipped with two metrics ρ and ρ_1. Let $x_0 \in X$. Now, we define a sequence $\{x_n\}$, where $x_n = T^n x_0$. If for some $n, x_n = x_{n+1} = x_{n+2}$ then x_n is a fixed point of T. Let $x_n \neq x_{n+1} \neq x_{n+2}$ for every $n \geq 0$. Now, from the given condition, we achieve $\min\{\rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a), \rho(x_{n-1}, x_n, a)\rho(x_{n-1}, x_n, a), \rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a) [\rho(x_{n-1}, x_n, a)]^2, \rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a)\} - \min\{\rho(x_{n-1}, x_n, a)\rho(x_{n-1}, x_n, a), \rho(x_{n-1}, x_{n+1}, a)\rho(x_n, x_n, a), \rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a), \rho(x_n, x_n, a)\rho(x_n, x_{n+1}, a)\} \leq r\rho(x_n, x_{n-1}, a) \rho(x_{n-1}, x_n, a) i.e. \min\{\rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a), [\rho(x_{n-1}, x_n, a)]^2, \rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a)[\rho(x_{n-1}, x_n, a)]^2, \rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a)\} - \min\{[\rho(x_{n-1}, x_n, a)]^2, \rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a)\} \leq r[\rho(x_{n-1}, x_n, a)]^2$ $i.e. \min\{\rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a), [\rho(x_{n-1}, x_n, a)]^2\} \leq r[\rho(x_{n-1}, x_n, a)]^2$. If $[\rho(x_{n-1}, x_n, a)]^2 \leq r[\rho(x_{n-1}, x_n, a)]^2$ then we have $q \geq 1$, which contradicts our assumption that $r \in (0,1)$. Hence $\rho(x_{n-1}, x_n, a)\rho(x_n, x_{n+1}, a) \leq r[\rho(x_{n-1}, x_n, a)]^2$. That means $\rho(x_n, x_{n+1}, a) \leq r\rho(x_{n-1}, x_n, a)$. Now, by symmetry we can proceed $\rho(x_n, x_{n+1}, a) \leq r^2\rho(x_{n-2}, x_{n-1}, a)$ similarly $\leq r^3\rho(x_{n-3}, x_{n-2}, a)$ and hence, $\leq r^n\rho(x_0, x_1, a)$. Now, on the other hand $\rho(x_n, x_{n+m}, a) \leq \rho(x_n, x_{n+m}, x_{n+1}) + \rho(x_n, x_{n+1}, a) + \rho(x_{n+1}, x_{n+m}, a) = \rho(x_n, x_{n+1}, x_{n+m}) + \rho(x_n, x_{n+1}, a) + \rho(x_{n+1}, x_{n+m}, a) = \rho(x_n, x_{n+1}, x_{n+m}) + \rho(x_n, x_{n+1}, a) + \rho (x_{n+1}, x_{n+m}, x_{n+2}) + \rho(x_{n+1}, x_{n+2}, a) + \rho(x_{n+2}, x_{n+m}, a) = \cdots i.e. \rho(x_n, x_{n+m}, a) \leq \rho(x_n, x_{n+1}, x_{n+m}) + \rho(x_n, x_{n+1}, a) + \rho(x_{n+1}, x_{n+m}, x_{n+2}) + \rho(x_{n+1}, x_{n+2}, a) + \cdots + \rho(x_{n+m-2}, x_{n+m-1}, x_{n+m}) + \rho(x_{n+m-1}, x_{n+m}, a) = \sum_{k=n}^{n+m} r^k\rho(x_0, x_1, x_{n+m}) + \sum_{k=n}^{n+m} r^k\rho(x_0, x_1, a)$. Now, since ρ is bounded that means $\rho(x, y, a) \leq N$ for some real number $N > 0$. Hence, we can achieve the $\rho(x_n, x_{n+m}, a) \leq 2N\sum_{k=n}^{n+m} r^k \leq 2N\frac{r^k}{1-r} \to 0$ when $n \to \infty$, because $r < 1$. This shows the resulting that $\{x_n\}$ is a Cauchy sequence in (X, ρ_1). Now, X is T-orbitally complete with respect to ρ_1. So, there exists $s \in X$ such that $\lim_{n\to\infty} x_n = s$. Again, T-orbitally continuous with respect to ρ_1 and this implies that $Ts = \lim_{n\to\infty} T^{n+1}x = s$. Hence s is a fixed point of T.

Example 2.3.1 If $X = \boldsymbol{R}_+$, and $r > 0$ be a constant and $\rho_1, \rho: X \times X \times X \to \boldsymbol{R}_+$ defined by $\rho(x, y, z) = [\max\{x, y\}]^r + |\max\{x, y\} - z|^r$ for all $x, y, z \in X$. Then (X, ρ) is a metric space with coefficient $n = 2r > 1$, but it is not a S-metric space. Indeed, for $x = 5$, $y = 2$, $z = 1$, $u = 4$ we have $\rho(x, y, z) = 5^r + 4^r$ and

$\rho(x,y,u)+\rho_1(y,y,u)+\rho_1(z,z,u)+\rho(u,u,u)=5^r+1+3^r+1+1+3^r-4^r=5^r+2\times 3^r+3-4^r$, hence $\rho(x,y,z)>\rho(x,y,u)+\rho_1(y,y,u)+\rho_1(z,z,u)+\rho(u,u,u)$ for all $r>0$. Hence, this is an example of bi two metric space.

Theorem 2.4 Let(X,ρ_h) be a complete quasi partial S_b-metric space with coefficient $t\geq 1$ and $T{:}\,X\to X$ be a mapping satisfying the following condition for all $\alpha,\beta,\gamma\in X$, $\rho_h(T\alpha,T\beta,T\gamma)\leq\lambda\max\{\rho_h(\alpha,\beta,\gamma),\rho_h(\alpha,\alpha,T\alpha),\rho_h(\beta,\beta,T\beta),\rho_h(\gamma,\gamma,T\gamma)\}$ where, $0\leq\lambda<\frac{1}{2s}$. Then T has a unique point $c\in X$ with $\rho_h(c,c,c)=0$.

Proof: Very first, we prove the uniqueness of fixed point of T if exists and c be a fixed point of T then $\rho_h(c,c,c)=0$. Let $x_1,x_2\in X$ be the two distinct fixed point of T. That means $Tx_1=x_1$ and $Tx_2=x_2$. Let us suppose that $\rho_h(x_1,x_1,x_1)>0$. Then from the given condition we have as follows $\rho_h(x_1,x_1,x_1)=\rho_h(Tx_1,Tx_1,Tx_1)<\lambda\rho_h(x_1,x_1,x_1)<\rho_h(x_1,x_1,x_1)$ and that shows a contradiction. Hence $\rho_h(x_1,x_1,x_1)=0$ by symmetry we can say that $\rho_h(x_2,x_2,x_2)=0$. Again, from the given condition $\rho_h(x_1,x_1,x_1)=\rho_h(Tx_1,Tx_1,Tx_1)\leq$ $\lambda\max\{\rho_h(x_1,x_1,x_2),\rho_h(x_1,x_1,\ Tx_1),\rho_h(x_2,x_2,Tx_2)\}=$ $\lambda\max\{\rho_h(x_1,x_1,x_2),\rho_h(x_1,x_1,x_1),\rho_h(x_2,x_2,x_2)\}=\lambda\rho_h(x_1,x_1,x_2)<$ $\rho_h(x_1,x_1,x_2)$. Therefore $\rho_h(x_1,x_1,x_2)=\rho_h(x_1,x_1,x_1)=\rho_h(x_2,x_2,x_2)$. Obviously, $x_1=x_2$.

Now, we shall prove the remaining part of the theorem. So assuming that $\vartheta_0\in X$ be an arbitrary and define a sequence $\{\vartheta_i\}$ by $\vartheta_{i+1}=T\vartheta_i$ for all $i\in\boldsymbol{N}\cup\{0\}$. Now, $i\in\boldsymbol{N}$ then from the given condition we achieve the following result $\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1})=\rho_h(T\vartheta_{i-1},T\vartheta_{i-1},T\vartheta_i)\leq$ $\lambda\max\{\rho_h(\vartheta_{i-1},\vartheta_{i-1},\vartheta_i),\rho_h(\vartheta_{i-1},\vartheta_{i-1},T\vartheta_{i-1}),\rho_h(\vartheta_i,\vartheta_i,T\vartheta_i)\}$ and this will be equal to $\lambda\max\{\rho_h(\vartheta_{i-1},\vartheta_{i-1},\vartheta_i),\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1})\}=\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1})$ which is a contradiction. Hence, $\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1})\leq\lambda\rho_h(\vartheta_{i-1},\vartheta_{i-1},\vartheta_i)$. in such a manner we can achieve $\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1})\leq\lambda^i\rho_h(\vartheta_0,\vartheta_0,\vartheta_1)$. So, for all $i\in\boldsymbol{N}$, $\rho_h(\vartheta_{i+1},\vartheta_{i+1},\vartheta_i)=\rho_h(T\vartheta_i,T\vartheta_i,T\vartheta_{i-1})\leq$ $\lambda\max\{\rho_h(\vartheta_i,\ \vartheta_i,\vartheta_{i-1}),\rho_h(\vartheta_i,\vartheta_i,T\vartheta_i),\rho_h(\vartheta_{i-1},\vartheta_{i-1},T\vartheta_{i-1})\}=$ $\lambda\max\{\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i-1}),\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1}),\rho_h\ (\vartheta_{i-1},\vartheta_{i-1},T\vartheta_{i-1})\}$. If $\max\{\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i-1}),\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1}),\rho_h(\vartheta_{i-1},\vartheta_{i-1},\vartheta_i)\}=\rho_h(\vartheta_i,\vartheta_i\ ,\vartheta_{i-1})$ then $\rho_h(\vartheta_{i+1},\vartheta_{i+1},\vartheta_i)\leq\lambda\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i-1})$. Hence, $\rho_h\left(\vartheta_{i+1},\vartheta_{i+1},\vartheta_i\right)\leq\lambda^i\rho_h(\vartheta_1,\vartheta_1,\vartheta_0)$. If $\max\{\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i-1}),\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1}),\rho_h(\vartheta_{i-1},\vartheta_{i-1},\vartheta_i)\}=\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1})$ then using the equation $\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1})\leq\lambda^i\rho_h(\vartheta_0,\vartheta_0,\vartheta_1)$ we get $\rho_h(\vartheta_{i+1},\vartheta_{i+1},\vartheta_i)\leq\lambda^{i+1}\rho_h(\vartheta_0,\vartheta_0,\vartheta_1)$. Similarly for the rest case $\rho_h(\vartheta_{i+1},\vartheta_{i+1},\vartheta_i)\leq\lambda^i\rho_h(\vartheta_0,\vartheta_0,\vartheta_1)$. For $i,j\in\boldsymbol{N}$ with $j>i$, $\rho_h\left(\vartheta_i,\vartheta_i,\vartheta_j\right)\leq t\left[2\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1})+\rho_h\left(\vartheta_j,\vartheta_j,\vartheta_{i+1}\right)\right]-\rho_h(\vartheta_{i+1},\vartheta_{i+1},\vartheta_{i+1})=2t\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1})+t\left[2t\rho_h\left(\vartheta_j,\vartheta_j,\vartheta_{i+2}\right)+t\rho_h(\vartheta_{i+1},\vartheta_{i+1},\vartheta_{i+2})-2\rho_h(\vartheta_{i+2},\vartheta_{i+2},\vartheta_{i+1})\right]=$ $2t\lambda^i\frac{1-(2t\lambda)^{j-i-1}}{1-2t\lambda}\max\{\rho_h(\vartheta_0,\vartheta_0,\vartheta_1),\rho_h(\vartheta_1,\vartheta_1,\vartheta_0)\}$. Passing through limits we have $\lim_{i,j\to\infty}\rho_h\ \rho_h\left(\vartheta_i,\vartheta_i,\vartheta_j\right)=0$. Thus $\{\vartheta_i\}$ is a Cauchy in (X,ρ_h). Since (X,ρ_h) is complete. So there exists $c \in X$ such that $\lim_{i\to\infty} \rho_h(\vartheta_i,\vartheta_i,c) = \lim_{i,j\to\infty} \rho_h\big(\vartheta_i,\vartheta_i,\vartheta_j\big) = \rho_h(c,c,c) = 0$ and this gives the result $\lim_{i\to\infty} \rho_h(c,c,\vartheta_i) = 0$. Lastly, we show that c is the fixed point of T. For all $i \in \boldsymbol{N}$. Now, $\rho_h(c,c,Tc) \leq t[2\rho_h(c,c,\vartheta_{i+1}) + \rho_h(Tc,Tc,T\vartheta_i)] - \rho_h(\vartheta_{i+1},\vartheta_{i+1},\vartheta_{i+1}) = t[2\rho_h(c,c\,\vartheta_{i+1}) + \lambda \max\{\rho_h(c,c,\vartheta_i),\rho_h(c,c,Tc),\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1})\}]$ that means $\rho_h(c,c,Tc) \leq \lambda t\rho_h(c,c,Tc) < \rho_h(c,c,Tc)$ which is a contradiction. $i.e. \rho_h(c,c,Tc) = 0$. Now, $\rho_h(c,c,Tc) \leq 3t\rho_h(Tc,Tc,T\vartheta_i) = 3t\lambda \max\{\rho_h(c,c,\vartheta_i),\rho_h(c,c,Tc),\rho_h(\vartheta_i,\vartheta_i,\vartheta_{i+1})\}$. Now, applying the limit $\rho_h(Tc,Tc,Tc) = 0$. Thus, $Tc = c$.

Example 2.4.1 If $X = \{0,1,2,3\}$ and $\rho_h(a,b,c) = |a-b|^2 + |b-c|^2 + |c-a|^2 + a$. Ofcourse, ρ_h is a quasi partial S_b-metric space on X with coefficient $t = 2$. (X,ρ_h) is not a partial S_b-metric space because $\rho_h(1,1,2) \neq \rho_h(2,2,1)$. Also (X,ρ_h) is neither a S-metric space, since $\rho_h(2,2,2) \neq 0$, nor a partial S-metric space because $\rho_h(0,0,3) > \rho_h(0,0,1) + \rho_h(0,0,1) + \rho_h(3,3,1) + \rho_h(1,1,1)$.

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