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Modified Version of Verma Measures of Information and their Kinship with Past Information Measures

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Abstract

A new parametric function, $V_a(P) = \sum_{i=1}^{n} \ln(1 + ap_i) - \sum_{i=1}^{n} p_i \ln p_i - \ln(1 + a),\ \ a > 0$ is proposed for the probability distribution, $P = (p_1, p_2, \dots \dots \dots, p_n)$ and its properties are studied. This function is the modified version of Verma $i.e.$ hybrid Shannon [9] information measures [11, 12]. In this paper the function which we have taken is twice differentiable and is used to obtain the corresponding measure of directed divergence, logistic type growth models and its measures in fuzzy set. We also maximize the proposed function through multivariate normal distribution.

Key Words: Measure of Information measures, Concave Function, Degenerate Distribution, Monotonic Increasing Function, Permutationally Symmetric Function, Continuous Function, Equality of Probabilities.

1. INTRODUCTION

In 1948, C. E. Shannon [9] gave the measure

$$S(P) = -\sum_{i=1}^{n} p_i \ln p_i \tag{1.1}$$

to measure its uncertainty or information measures. It can also be regarded as a measure of equality of probabilities $p_1, p_2 \dots \dots \dots, p_n$ among themselves.

Later in, 1972, J. P. Burg [1], 1986, J. N. Kapur [4], [7] and in 2012, R. K. Verma [11, 12] gave the measures,

$$B(P) = \sum_{i=1}^{n} \ln p_i, \tag{1.2}$$

$$K_a(P) = -\sum_{i=1}^{n} p_i \ln p_i + \frac{1}{a}\sum_{i=1}^{n} (1 + ap_i) \ln(1 + ap_i) - \frac{1}{a}(1 + a) \ln(1 + a),\ a > 0 \tag{1.3}$$

and,

$$V_a(P) = \sum_{i=1}^{n} \ln(1 + ap_i) - \sum_{i=1}^{n} \ln p_i - \ln(1 + a),\ a > 0 \qquad (1.4)$$

Shannon's and Burg's measures do not have any parameter, while Kapur's measure has one parameter. When it is maximized by Lagrange's method, subject to linear constraints on probabilities, then the measures due to Shannon, Burg and Kapur always give non-negative probabilities. Shannon's measure has been the most successful and most widely used measure. Burg's measure has also been successful, but it is always negative and as such it is hard to interpret it as a measure of uncertainty. However, it can be used for information measures maximization purposes and it has been so used [5]. Moreover, its maximum value decreases with n but, this is not a desirable condition for a measure of information measures.

Since 1996, Kapur [6] was introduced the fuzzification, of these two pre-described information measures (1.2) and (1.3), in his original book. He gave the measures in a fuzzy set as follows:

$$K_a(A) = -\sum_{i=1}^{n} \left[\mu_A(x_i) \ln \mu_A(x_i) + \left(1 - \mu_A(x_i)\right) \ln\left(1 - \mu_A(x_i)\right)\right] + \frac{1}{a}\sum_{i=1}^{n} \left(1 + a\mu_A(x_i)\right) \ln\left(1 + a\mu_A(x_i)\right) + \frac{1}{a}\sum_{i=1}^{n} \left(1 + a - a\mu_A(x_i)\right) \ln\left(1 + a - a\mu_A(x_i)\right) - \frac{1}{a}(1 + a)\ln(1 + a),\ a > 0 \qquad (1.5)$$

and,

$$B(A) = \sum_{i=1}^{n} \ln \mu_A(x_i) + \sum_{i=1}^{n} \ln\left(1 - \mu_A(x_i)\right). \qquad (1.6)$$

Similarly in 2012, R. K. Verma [8] gave the fuzzified form of his own information measure $i.e.$ (1.4) as follows,

$$V_a(A) = -\sum_{i=1}^{n} \ln\left(1 - \mu_A(x_i)\right) - \sum_{i=1}^{n} \ln\left(1 + a - \mu_A(x_i)\right) + \sum_{i=1}^{n} \ln \mu_A(x_i) + \sum_{i=1}^{n} \ln\left(1 - \mu_A(x_i)\right) + \ln(1 + a),\ a > 0. \qquad (1.7)$$

In this paper we attempt to modify the entropy due to Verma [11], to obtained a new measure of information, with weaker and flexible parameter a, which provides us the greater flexibility to develop some new ideas in this field.

2. NEW RESULTS

2.1 SOME PROPERTIES OF MODIFIED VERSION OF VERMA [11] *i.e.* HYBRID SHANNON [9] INFORMATION MEASURES

The measure is defined by,

$$V_a(P) = \sum_{i=1}^{n} \ln(1 + ap_i) - \sum_{i=1}^{n} p_i \ln p_i - \sum_{i=1}^{n} \ln(1 + a)p_i\ ,\ a > 0 \qquad (2.1.1)$$

It has the following properties:

(1) It is a continuous function of $p_1, p_2, \dots\dots\dots, p_n$, so that it changes by a small

amount when $p_1, p_2 \dots \dots \dots p_n$ change by small amounts.

(2) It is a permutationally symmetric function of $p_1, p_2 \dots \dots \dots, p_n$ *i.e.* the function does not change when $p_1, p_2, \dots \dots \dots, p_n$ are permuted among themselves.

(3) It is maximum subject to natural constraints $\sum_{i=1}^{n} p_i = 1$ when

$$p_1 = p_2 = \dots \dots \dots = p_n = \frac{1}{n} \tag{2.1.2}$$

(4) The maximum value is an increasing function of n. In fact the maximum value is given by,

$$f(n) = n \ln\left(1 + \frac{a}{n}\right) - \ln\frac{1}{n} - \ln(1 + a), \tag{2.1.3}$$

so that,

$$f'(n) = \ln\left(1 + \frac{a}{n}\right) - \frac{\frac{a}{n}}{\left(1+\frac{a}{n}\right)} + \frac{1}{n} \tag{2.1.4}$$

and,

$$f''(n) = -\frac{1}{n^2} - \frac{a^2}{n(n+a)^2} < 0 \tag{2.1.5}$$

so that, $f(n)$ is a concave function of n.

Now,

$$f'(n) = \ln\left(1 + \frac{a}{n}\right) - \frac{\frac{a}{n}}{\left(1 + \frac{a}{n}\right)} + \frac{1}{n}$$

$$= \ln\left(1 + \frac{a}{n}\right) - 1 + \frac{1}{1 + \frac{a}{n}} + \frac{1}{n}$$

$$= \ln y - 1 + \frac{1}{y} + \frac{1}{n};\ y = 1 + \frac{a}{n} > 0$$

$$= \frac{ny \ln y - ny + n + y}{ny} > 0 \tag{2.1.6}$$

Since, $y > 0$ and $ny \ln y - ny + n + y > 0$ when $y \geq 0$,
we get,

$$f'(n) > 0, \tag{2.1.7}$$

so that, $f(n)$ is a monotonic increasing function of n.

Now,

$$\frac{\partial}{\partial p_i} V_a(P) = \frac{a}{1+ap_i} - 1 - \ln p_i,\ \frac{\partial^2}{\partial p_i^2} V_a(P) = -\frac{1+a^2p_i+a^2p_i^2+2ap_i}{p_i(1+ap_i)^2}$$

and,

$$\frac{\partial^2}{\partial p_i \partial p_j} V_a(P) = 0 \tag{2.1.8}$$

so that, $V_a(P)$ is a concave function of $p_1, p_2, \dots \dots \dots, p_n$.

Since $V_a(P)$ is a concave function and its domain is

$$p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, \dots \dots \dots, p_n \geq 0\ ;\ \sum_{i=1}^{n} p_i = 1, \tag{2.1.9}$$

Its minimum value occurs at each of the degenerate distributions

$$\Delta_i = (0,0 \dots \dots \dots, 1,0,0, \dots \dots \dots, 0)\ ; i = 1,2, \dots \dots \dots, n \tag{2.1.10}$$

where in Δ_i , unity occurs in the i^{th} place and for each of these, its value is zero. Thus,

$$V_a(P) \geq 0 \tag{2.1.11}$$

and it vanishes only when P concides with one of the degenerate distributions, *i.e.*

when there is perfect certainty and uncertainty is zero.
Let $p_i < p_j$ and let us increase p_i by x and decrease p_j by x so that, $p_i + x$ is still $\leq p_j - x$, then

$$V_a(P') - V_a(P) = \ln(1 + a(p_i + x)) + \ln\left(1 + a(p_j - x)\right) - (p_i + x)\ln(p_i + x) - (p_j - x)\ln(p_j - x) + \ln(1 + a)(p_i + x + p_j - x) - \ln(1 + ap_i) - \ln(1 + ap_j) + p_i \ln p_i + p_j \ln p_j + \ln(1 + a)p_i + \ln(1 + a)p_j = h(x), \text{ say} \quad (2.1.12)$$

so that, $h'(x) = \frac{a}{1+ap_i+ax} - \frac{a}{1+ap_j-ax} -$

$$(p_i + x)\frac{1}{(p_i + x)} - \ln(p_i + x) + (p_j - x)\frac{1}{p_j - x} + \ln(p_j - x) \quad (2.1.13)$$

$$= \frac{a}{1+ap_i+ax} - \frac{a}{1+ap_j-ax} - \ln(p_i + x) + \ln(p_j - x) \quad (2.1.14)$$

and,

$$h''(x) = -\frac{a^2}{(1+ap_i+ax)^2} - \frac{a^2}{(1+ap_j-ax)^2} - \frac{1}{p_i+x} - \frac{1}{p_j-x} < 0, \quad (2.1.15)$$

so that, $h(x)$ will be maximum when,

$$p_i + x = p_j - x. \quad (2.1.16)$$

Thus, as the probabilities become more and more equal, the measure increases, so that the measure can be used as a measure of equality of probabilities.
Although the measure $V_a(P)$ is inspired by Kapur [4] and Burg [1] entropy measures and this measure suffer from the weakness of Burg's measure, but it satisfies all the important properties satisfied by Shannon's [9] measure of information measures except additivity and recursivity. However, these properties are unimportant for information measures maximization purpose and hence, $V_a(P)$ can be used as a measure of uncertainty or information measures.

2.2 CONCAVITY OF MODIFIED VERMA [11] *i.e.* HYBRID SHANNON [9] INFORMATION MEASURES V_{max} WHEN THE MEAN IS PRESCRIBED

Maximizing,

$$V_a(P) = \sum_{i=1}^{n} \ln(1 + ap_i) - \sum_{i=1}^{n} p_i \ln p_i - \ln(1 + a)\ , a > 0$$

subject to some constraints,

$$\sum_{i=1}^{n} p_i = 1 \quad (2.2.1)$$

and,

$$\sum_{i=1}^{n} ip_i = m. \quad (2.2.2)$$

Suppose that,

$$p_i + \frac{1}{a} = \frac{1}{\lambda+\mu i} \quad (2.2.3)$$

so that,

$$\sum_{i=1}^{n} \frac{1}{\lambda+\mu i} = 1 + \frac{n}{a} \quad (2.2.4)$$

and,

$$\sum_{i=1}^{n} \frac{i}{\lambda+\mu i} = m + \frac{n(n+1)}{2a} \tag{2.2.5}$$

Now,

$$\lambda\left(1+\frac{n}{a}\right) + \mu\left(m + \frac{n(n+1)}{2a}\right) = n \tag{2.2.6}$$

or,

$$\lambda = \frac{n-\mu\left(m+\frac{n(n+1)}{2a}\right)}{1+\frac{n}{a}} \tag{2.2.7}$$

and,

$$\lambda + \mu i = \frac{n-\mu\left(m+\frac{n(n+1)}{2a}\right)}{1+\frac{n}{a}} + \mu i \tag{2.2.8}$$

$$= \frac{n+\mu\left\{\left(1+\frac{n}{a}\right)i - m - \frac{n(n+1)}{2a}\right\}}{1+\frac{n}{a}} \tag{2.2.9}$$

Also,

$$p_i = \frac{a+n}{na+\mu\left\{(n+a)i - ma - \frac{n(n+1)}{2}\right\}} - \frac{1}{a} \tag{2.2.10}$$

$$= \frac{a-\mu\left\{\left(1+\frac{n}{a}\right)i - m - \frac{n(n+1)}{2a}\right\}}{na+\mu\left\{(n+a)i - ma - \frac{n(n+1)}{2}\right\}}$$

where μ is determined from the equation

$$\frac{n}{a} + 1 = \sum_{i=1}^{n} \frac{a+n}{na+\mu\left\{(n+a)i - ma - \frac{n(n+1)}{2}\right\}}$$

or,

$$\frac{1}{a} = \sum_{i=1}^{n} \frac{1}{na+\mu\left\{(n+a)i - ma - \frac{n(n+1)}{2}\right\}} \tag{2.2.11}$$

Now,

$$\mathrm{V}_{\max} = \sum_{i=1}^{n} \ln(1+ap_i) - \sum_{i=1}^{n} p_i \ln p_i - \ln(1+a)$$

$$= \sum_{i=1}^{n} \ln\frac{a}{\lambda+\mu i} - \sum_{i=1}^{n}\left(\frac{1}{\lambda+\mu i} - \frac{1}{a}\right)\ln\left(\frac{1}{\lambda+\mu i} - \frac{1}{a}\right) - \ln(1+a)$$

$$= n\ln a + \sum_{i=1}^{n} \ln(\lambda+\mu i) - \sum_{i=1}^{n}\left(\frac{1}{\lambda+\mu i} - \frac{1}{a}\right)\ln\left(\frac{1}{\lambda+\mu i} - \frac{1}{a}\right) - \ln(1+a)$$

$$= \ln\frac{a^n}{1+a} - \sum_{i=1}^{n} \ln\left[\frac{n+\mu\left\{\left(1+\frac{n}{a}\right)i - m - \frac{n(n+1)}{2a}\right\}}{1+\frac{n}{a}}\right] -$$

$$\sum_{i=1}^{n}\left[\frac{a-\mu\left\{\left(1+\frac{n}{a}\right)i - m - \frac{n(n+1)}{2a}\right\}}{na+\mu\left\{(n+a)i - ma - \frac{n(n+1)}{2}\right\}}\right]\ln\left[\frac{a-\mu\left\{\left(1+\frac{n}{a}\right)i - m - \frac{n(n+1)}{2a}\right\}}{na+\mu\left\{(n+a)i - ma - \frac{n(n+1)}{2}\right\}}\right]$$

$$= \ln\frac{a^n}{1+a} - \sum_{i=1}^{n} \ln\left[na+\mu\left\{(n+a)i - ma - \frac{n(n+1)}{2}\right\}\right] + \sum_{i=1}^{n} \ln(n+a)$$

$$-\sum_{i=1}^{n}\left[\frac{a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}}{na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}}\right]\ln\left[a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}\right]+$$
$$\sum_{i=1}^{n}\left[\frac{a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}}{na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}}\right]\ln\left[na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}\right].$$

Thus,

$$\mathrm{V}_{\max}=\ln\frac{a^n}{1+a}+\ln(n+a)^n-\sum_{i=1}^{n}\ln\left[na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}\right]$$
$$-\sum_{i=1}^{n}\left[\frac{a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}}{na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}}\right]\ln\left[a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}\right]+$$
$$\sum_{i=1}^{n}\left[\frac{a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}}{na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}}\right]\ln\left[na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}\right] \tag{2.2.12}$$

Now,

$$\frac{d\mathrm{V}_{\max}}{dm}=-\sum_{i=1}^{n}\frac{1}{na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}}(-\mu a)-$$
$$\sum_{i=1}^{n}\frac{a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}}{na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}}.\frac{1}{a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}}.\mu-$$
$$\sum_{i=1}^{n}\ln\left[a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}\right]\frac{\left[na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}\right]\mu-\left[a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}\right](-\mu a)}{\left[na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}\right]^2}$$
$$+\sum_{i=1}^{n}\frac{a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}}{na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}}.\frac{1}{na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}}.(-\mu a)+$$
$$\sum_{i=1}^{n}\ln\left[na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}\right]\frac{\left[na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}\right]\mu-\left[a-\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}\right](-\mu a)}{\left[na+\mu\left\{(n+a)i-ma-\frac{n(n+1)}{2}\right\}\right]^2}$$

But,

$$\sum_{i=1}^{n}\left(1+\frac{n}{a}\right)i=\sum_{i=1}^{n}\frac{i}{\lambda+\mu i}=m+\frac{n(n+1)}{2a}$$

so that,

$$\frac{d\mathrm{V}_{\max}}{dm}=\left[1-\left(\frac{1}{a}+\frac{1}{n}\right)(1-\ln n)\right]\mu \tag{2.2.13}$$

and

$$\frac{d^2\mathrm{V}_{\max}}{dm^2}=\left[1-\left(\frac{1}{a}+\frac{1}{n}\right)(1-\ln n)\right]\frac{d\mu}{dm} \tag{2.2.14}$$

and, from (2.2.9) we have

$$n+\mu\left\{\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right\}=1$$

so that,

$$\frac{d\mu}{dm} = \frac{\mu}{\left(1+\frac{n}{a}\right)i - m - \frac{n(n+1)}{2a}}$$

$$= \frac{\mu \sum_{i=1}^{n}\left(p_i+\frac{1}{a}\right)^2}{\sum_{i=1}^{n}\left(p_i+\frac{1}{a}\right)^2\left[\left(1+\frac{n}{a}\right)i-m-\frac{n(n+1)}{2a}\right]} \qquad (2.2.15)$$

V_{max} will be a concave function of m if $\frac{d\mu}{dm} < 0$ and $1-\left(\frac{1}{a}+\frac{1}{n}\right)(1-\ln n) > 0$ $i.e.$ if either $\mu > 0$, denominater < 0 or $\mu < 0$, denominater > 0. Also V_{max} will be a concave function of m if $\frac{d\mu}{dm} > 0$ and $1-\left(\frac{1}{a}+\frac{1}{n}\right)(1-\ln n) < 0$ $i.e.$ if either $\mu < 0$, denominater > 0 or $\mu > 0$, denominater < 0.

Case I. If $\mu > 0, m < \frac{n+1}{2}$, probabilities are decreasing, and

Case II. If $\mu < 0, m > \frac{n+1}{2}$, probabilities are increasing.

Thus, we have proved that for all positive values of m, V_{max} is a concave function of m.

2.3 LIMITING CASE OF MODIFIED VERMA [11, 12] $i.e.$ HYBRID SHANNON [9] ENTROPY

When we apply the limiting case $a \to 0$ in (2.1.1) then we get Shannon's entropy $i.e.$

$$S(P) = -\sum_{i=1}^{n} p_i \ln p_i$$

Renyi's entropy [8] $i.e.$

$$R(P) = \frac{1}{1-a}\ln\sum_{i=1}^{n} p_i^a$$

Havrda-Charvat entropy [2] $i.e.$

$$H(P) = \frac{1}{1-a}\left[\sum_{i=1}^{n} p_i^a - 1\right]$$

which is also same as Tsallis's [10] entropy.

CONCLUSION:

The proposed function satisfies all the important properties satisfied by Shannon's [9] entropy. In the proposed function, it has the greater flexibility in applications due to the presence of weaker and flexible parameter a. So whenever some modifications are imposed or some limiting conditions are applied then we carried out some important and interesting results which may be useful in the domain of the channel capacity in wired and wireless communication system in the presence of noise.

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