

OSCILLATION CRITERIA FOR NONLINEAR FUNCTIONAL NEUTRAL DELAY DIFFERENCE EQUATIONS WITH VARIABLE CO-EFFICIENTS

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ABSTRACT

In this paper, some new sufficient conditions for the oscillation of all solutions of first order nonlinear neutral delay difference equations with variable coefficients of the form

$$\Delta[r(n)(a(n)x(n) + p(n)x(n - \tau))] + q(n)f(x(n - \sigma)) = 0, \quad n \geq n_0; \quad (*)$$

are established. Our obtained results extending and / (or) improving some well known results in the literature. Our main results are obtained by using some inequality techniques. Some illustrative examples are also included.

AMS Subject Classification 2010: 39A10, 39A12.

Keywords and Phrases: Oscillation, nonoscillation, nonlinear, neutral, delay difference equations, variable co-efficients.

1. INTRODUCTION

In this paper, we consider the following first order neutral delay difference equations with variable coefficients of the form

$$\Delta[r(n)(a(n)x(n) + p(n)x(n - \tau))] + q(n)f(x(n - \sigma)) = 0, \quad n \geq n_0; \quad (1)$$

where Δ is the forward difference operator defined by $\Delta x(n) = x(n + 1) - x(n)$, $\{r(n)\}$, $\{a(n)\}$, $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive real numbers defined on $N(n_0) = \{n_0, n_0 + 1, \dots\}$.

Throughout this paper the following conditions are assumed to be hold.

- (i) there exists constants a_0, p_0 and p_1 such that $a(n) \leq a_0$ and $p_0 \leq p(n) \leq p_1 < \infty$;
- (ii) τ and σ are positive integers;
- (iii) $f: R \rightarrow R$ is a continuous function satisfies $uf(u) > 0$ for $u \neq 0$ and there exists a positive constant k_0 such that

$$\frac{f(u)}{u} \geq k_0 > 0.$$

Let us choose a positive integer $n^* > \max\{\tau, \sigma\}$. By a solution of (1) on $N(n_0) = \{n_0, n_0 + 1, \dots\}$, we mean a real sequence $\{x(n)\}$ which is defined on $n \geq n_0 - n^*$ and which satisfies (1) for $n \in N(n_0)$. A solution $\{x(n)\}$ of (1) on $N(n_0)$ is said to be oscillatory if for every positive integers $N_0 > n_0$ there exists $n \geq N_0$ such that $x(n)x(n + 1) \leq 0$, otherwise $\{x(n)\}$ is said to be nonoscillatory.

The oscillatory behavior of difference equations has been intensively studied in recent years. Most of the literature has been concerned with equation of type (1) with $r(n) = 1$ and $a(n) = 1$ (see [1-10] and reference cited there in). But very little is known regarding the oscillation of first order neutral delay difference equations similar to (1). The purpose of this paper is to study the oscillatory properties of (1).

In the sequel, unless otherwise specified, when we write a functional inequality we shall assume that it holds for all sufficiently large value of n .

2. AUXILIARY LEMMAS

In this section, we give some useful lemmas which will play an important role in the study of the oscillation of (1).

Lemma 2.1. [4] *Assume that $\{q(n)\}$ is a sequence of positive real numbers and σ is a positive integer. Then the difference inequality*

$$\Delta x(n) + q(n)x(n - \sigma) \leq 0, \quad n \geq n_0 \quad (2)$$

has an eventually positive solution if and only if the equation

$$\Delta x(n) + q(n)x(n - \sigma) = 0, \quad n \geq n_0 \quad (3)$$

has an eventually positive solution.

Lemma 2.2. Assume that

$$\limsup_{n \rightarrow \infty} \sum_{s=n}^{n+\sigma} q(s) > 0. \quad (4)$$

If $\{x(n)\}$ is an eventually positive solution of the delay difference equation

$$\Delta x(n) + q(n)x(n - \sigma) = 0, \quad n \geq n_0, \quad (5)$$

then

$$\liminf_{n \rightarrow \infty} \frac{x(n - \sigma)}{x(n)} < \infty. \quad (6)$$

Proof. In view of the assumption that there exist a constant $d > 0$ and a sequence $\{n_k\}$ of integers such that $n_k \rightarrow \infty$ as $k \rightarrow \infty$ and

$$\sum_{s=n_k}^{n_k+\sigma} q(s) \geq d, \quad k = 1, 2, 3, \dots$$

Then there exists $\xi_k \in \{n_k, n_{k+1}, \dots, n_{k+\sigma}\}$ for every k such that

$$\sum_{s=n_k}^{\xi_k} q(s) \geq \frac{d}{2} \quad \text{and} \quad \sum_{s=\xi_k}^{n_k+\sigma} q(s) \geq \frac{d}{2}. \quad (7)$$

Summing the equation (6) for n_k to ξ_k and ξ_k to $n_k + \sigma$, we find

$$x(\xi_k + 1) - x(n_k) + \sum_{s=n_k}^{\xi_k} q(s) x(s - \sigma) = 0 \quad (8)$$

and

$$x(n_k - \sigma + 1) - x(n_k) + \sum_{s=\xi_k}^{n_k+\sigma} q(s) x(s - \sigma) = 0. \quad (9)$$

By omitting the first terms in (9) and (10) and by using the decreasing nature of $\{x(n)\}$ and (7), we find

$$-x(n_k) + \frac{d}{2} x(\xi_k - \sigma) \leq 0$$

and

$$-x(\xi_k) + \frac{d}{2}x(n_k) \leq 0$$

or

$$\frac{x(\xi_k - \sigma)}{x(\xi_k)} \leq \left(\frac{2}{d}\right)^2.$$

This completes the proof.

Lemma 2.3. Assume that $\sigma > \tau$, $r(n) \equiv 1$, and

$$\limsup_{n \rightarrow \infty} \sum_{s=n}^{n+\sigma-\tau} q(s) > 0. \quad (10)$$

Let $\{x(n)\}$ be an eventually positive solution of the equation (1). Set

$$z(n) = a(n)x(n) + p(n)x(n - \tau). \quad (11)$$

Then

$$\liminf_{n \rightarrow \infty} \frac{z(n - \sigma + \tau)}{z(n)} < \infty. \quad (12)$$

Proof. In view of (1) and (12), we have $z(n) > 0$ eventually and decreasing. From (11), we get

$$z(n + \tau) \geq a(n + \tau)x(n + \tau) + p_0x(n) \quad (13)$$

and

$$p(n)x(n - \tau) = z(n) - a(n)x(n) \geq z(n) - a_0x(n). \quad (14)$$

Since $\{z(n)\}$ is decreasing, we have $z(n) > z(n + \tau)$ which implies with (13) that

$$z(n) \geq p_0x(n)$$

or

$$x(n) \leq \frac{z(n)}{p_0}.$$

Using the above inequality in (14), we obtain

$$p(n)x(n - \tau) \geq z(n) - \frac{a_0z(n)}{p_0},$$

or

$$p_1x(n - \tau) \geq \frac{z(n)(p_0 - a_0)}{p_0},$$

or

$$x(n - \tau) \geq \frac{z(n)(p_0 - a_0)}{p_1 p_0}.$$

Hence

$$x(n - \sigma) \geq \frac{(p_0 - a_0)}{p_1 p_0} z(n + \tau - \sigma). \quad (15)$$

From (1) and (15), we get

$$\Delta z(n) + \frac{k_0(p_0 - a_0)}{p_1 p_0} q(n) z(n + \tau - \sigma) \leq 0. \quad (16)$$

By Lemma 2.1, we find that the equation

$$\Delta z(n) + \frac{k_0(p_0 - a_0)}{p_1 p_0} q(n) z(n + \tau - \sigma) = 0 \quad (17)$$

has an eventually positive solution as well.

As a result, by Lemma 2.2 and (10), we have

$$\liminf_{n \rightarrow \infty} \frac{z(n + \tau - \sigma)}{z(n)} < \infty,$$

which is the desired result.

The proof is complete.

Lemma 2.4. Assume that $\sigma > \tau$, and $r(n) \equiv 1$. If (1) has an eventually positive solution, then

$$\sum_{s=n}^{n+\sigma-\tau} q(s) \leq \frac{p_1 p_0}{k_0(p_0 - a_0)} \quad (18)$$

for all sufficiently large n .

Proof. Proceeding as in the proof of Lemma 2.3, we again obtain the inequality (16). Summing (16) from n to $n + \sigma - \tau$, we get

$$z(n - \tau + \sigma + 1) - z(n) + \frac{k_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n}^{n+\sigma-\tau} q(s) z(s - \sigma + \tau) \leq 0.$$

Using the decreasing nature of $\{z(n)\}$, we get

$$z(n - \tau + \sigma + 1) - z(n) + \frac{k_0(p_0 - a_0)}{p_1 p_0} z(n) \sum_{s=n}^{n+\sigma-\tau} q(s) \leq 0.$$

Then

$$z(n - \tau + \sigma + 1) + \left(\frac{k_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n}^{n+\sigma-\tau} q(s) - 1 \right) z(n) \leq 0.$$

Hence for all sufficiently large n , we get

$$\sum_{s=n}^{n+\sigma-\tau} q(s) \leq \frac{p_1 p_0}{k_0(p_0 - a_0)},$$

which is the desired result. The proof is completed.

3. OSCILLATION OF SOLUTIONS

Theorem 3.1. Assume that $\sigma > \tau$ and $r(n) \equiv 1$, and (11) hold. If

$$\sum_{n=n_0}^{\infty} q(n) \ln \left(\frac{e k_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} q(s) \right) = \infty. \quad (19)$$

Then every solution of (1) oscillates.

Proof. Assume the contrary. Without loss of generality we may assume that $\{x(n)\}$ is an eventually positive solution of (1). Set $z(n)$ as in (11). Then $\{z(n)\}$ is eventually positive and decreasing. Also $\{x(n)\}$ satisfies the inequality (16). That is,

$$\Delta z(n) + \frac{k_0(p_0 - a_0)}{p_1 p_0} q(n) z(n + \tau - \sigma) \leq 0.$$

Define the sequence $\{u(n)\}$ as

$$u(n) = \frac{-\Delta z(n)}{z(n)}.$$

Then $\{u(n)\}$ is eventually nonnegative. So, there exists $n_1 \geq n_0$ with $z(n_1) > 0$. We can easily show that

$$\Delta z(n) \leq z(n_1) \exp \left(- \sum_{s=n_1}^{n-1} q(s) \right).$$

Moreover, $\{u(n)\}$ satisfies

$$u(n) \geq \frac{k_0(p_0 - a_0)}{p_1 p_0} q(n) \exp \left(\sum_{s=n+\tau-\sigma}^{n-1} u(s) \right). \quad (20)$$

By using the inequality

$$e^{rx} \geq x + \frac{\ln(er)}{r}, \quad x, r > 0,$$

we have from (20),

$$\begin{aligned} u(n) &\geq \frac{k_0(p_0 - a_0)}{p_1 p_0} q(n) \exp\left(\frac{C(n)}{C(n)} \sum_{s=n+\tau-\sigma}^{n-1} u(s)\right) \\ &\geq \frac{k_0(p_0 - a_0)}{p_1 p_0} q(n) \left(\frac{1}{C(n)} \sum_{s=n+\tau-\sigma}^{n-1} u(s) + \frac{\ln(eC(n))}{C(n)}\right), \end{aligned}$$

where

$$C(n) = \frac{k_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} q(s).$$

Therefore,

$$u(n) \sum_{s=n+1}^{n+\sigma-\tau} q(s) - q(n) \sum_{s=n+\tau-\sigma}^{n-1} u(s) \geq q(n) \ln \left[\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} q(s) \right].$$

Hence for $\xi > N + \sigma - \tau$

$$\begin{aligned} \sum_{n=N}^{\xi-1} u(n) \left[\sum_{s=n+1}^{n+\sigma-\tau} q(s) \right] - \sum_{n=N}^{\xi-1} q(n) \left(\sum_{s=n+\tau-\sigma}^{n-1} u(s) \right) \\ \geq \sum_{n=N}^{\xi-1} q(n) \ln \left(\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} q(s) \right). \end{aligned} \quad (21)$$

By interchanging the order of summation, we have

$$\sum_{n=N}^{\xi-1} q(n) \sum_{s=n+\tau-\sigma}^{n-1} u(s) \geq \sum_{n=N}^{\xi+\tau-\sigma-1} u(n) \sum_{s=n+1}^{n-\tau+\sigma} q(s). \quad (22)$$

Combining (21) and (22) leads to

$$\sum_{n=\xi+\tau-\sigma}^{\xi-1} u(n) \sum_{s=n+1}^{n-\tau+\sigma} q(s) \geq \sum_{n=N}^{\xi-1} q(n) \ln \left[\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} q(s) \right]. \quad (23)$$

Using (18) of Lemma 2.4 in (23), we obtain

$$\sum_{n=\xi+\tau-\sigma}^{\xi-1} u(n) \geq \frac{k_0(a_0 - p_0)}{p_1 p_0} \sum_{n=N}^{\xi-1} q(n) \ln \left(\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} q(s) \right),$$

or

$$\ln \frac{z(\xi + \tau - \sigma)}{z(n)} \geq \frac{k_0(p_0 - a_0)}{p_1 p_0} \sum_{n=N}^{\xi-1} q(n) \ln \left[\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} q(s) \right].$$

This result along with condition (20) leads to

$$\lim_{n \rightarrow \infty} \frac{z(n + \tau - \sigma)}{z(n)} = \infty,$$

which contradicts (12) and completes the proof.

Theorem 3.2. Assume that $\sigma > \tau$, and $\Delta r(n) > 0$. If

$$0 < C \leq \liminf_{n \rightarrow \infty} \sum_{s=n}^{n+\sigma-\tau} \frac{q(s)}{r(s - \sigma + \tau)} \quad (24)$$

and

$$\sum_{n=n_0}^{\infty} \frac{q(n)}{r(n + \tau - \sigma)} \ln \left(\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} \frac{q(s)}{r(s + \tau - \sigma)} \right) = \infty. \quad (25)$$

Then every solution of (1) oscillates.

Proof. For the sake of obtaining a contradiction, without loss of generality, we assume that $\{x(n)\}$ is an eventually positive solution of (1). Set $z(n)$ as in (11). Then $\{z(n)\}$ is positive and decreasing. Proceeding as in the proof of Lemma 2.3, we get

$$\Delta(r(n)z(n)) + \frac{k_0(p_0 - a_0)}{p_1 p_0} q(n)z(n + \tau - \sigma) \leq 0. \quad (26)$$

Set

$$y(n) = r(n)z(n) \quad (27)$$

using this in (26), we get

$$\Delta y(n) + \frac{k_0(p_0 - a_0)}{p_1 p_0} \frac{q(n)}{r(n + \tau - \sigma)} y(n + \tau - \sigma) \leq 0. \quad (28)$$

Set

$$\lambda(n) = -\frac{\Delta y(n)}{y(n)}. \quad (29)$$

Then $\lambda(n) > 0$ eventually and $(\lambda(n))$ satisfies the inequality

$$\lambda(n) \geq \frac{k_0(p_0 - a_0)}{p_1 p_0} \frac{q(n)}{r(n + \tau - \sigma)} \exp\left(\sum_{s=n+\tau-\sigma}^{n-1} \lambda(s)\right). \quad (30)$$

Applying the inequality

$$e^{rx} \geq x + \frac{\ln(er)}{r}, \quad x, r > 0,$$

to (30) yields that

$$\begin{aligned} \lambda(n) &\geq \frac{k_0(p_0 - a_0)}{p_1 p_0} \left(\frac{q(n)}{r(n + \tau - \sigma)}\right) \exp\left(\frac{D(n)}{D(n)} \sum_{s=n+\tau-\sigma}^{n-1} \lambda(s)\right) \\ &\geq \left(\frac{k_0(p_0 - a_0)}{p_1 p_0}\right) \left(\frac{q(n)}{r(n + \tau - \sigma)}\right) \left(\frac{1}{D(n)} \sum_{s=n+\tau-\sigma}^{n-1} \lambda(s) + \frac{\ln(eD(n))}{D(n)}\right), \end{aligned}$$

where

$$D(n) = \frac{k_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n-\tau+\sigma} \frac{q(s)}{r(s + \tau - \sigma)}. \quad (31)$$

Therefore,

$$\begin{aligned} \lambda(n) &\sum_{s=n+1}^{n-\tau+\sigma} \frac{q(s)}{r(s + \tau - \sigma)} - \frac{q(n)}{r(n + \tau - \sigma)} \sum_{s=n+\tau-\sigma}^{n-1} \lambda(s) \\ &\geq \frac{q(n)}{r(n + \tau - \sigma)} \ln\left(\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n-\tau+\sigma} \frac{q(s)}{r(s + \tau - \sigma)}\right). \end{aligned}$$

Hence, for $\eta > N + \sigma - \tau$

$$\begin{aligned} &\sum_{n=N_1}^{\eta-1} \lambda(n) \sum_{s=n+1}^{n-\tau+\sigma} \frac{q(s)}{r(s + \tau - \sigma)} - \sum_{n=N_1}^{\eta-1} \frac{q(n)}{r(n + \tau - \sigma)} \sum_{s=n+\tau-\sigma}^{n-1} \lambda(s) \\ &\geq \sum_{n=N_1}^{\eta-1} \frac{q(n)}{r(n + \tau - \sigma)} \ln\left(\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} \frac{q(s)}{r(s + \tau - \sigma)}\right). \quad (32) \end{aligned}$$

By interchanging the order of summation, we have

$$\sum_{n=N_1}^{\eta-1} \frac{q(n)}{r(n+\tau-\sigma)} \left(\sum_{s=n+\tau-\sigma}^{n-1} \lambda(s) \right) \geq \sum_{n=N_1}^{n+\tau-\sigma-1} \lambda(n) \sum_{s=n+1}^{n-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)}. \quad (33)$$

From (32) and (33), we have

$$\begin{aligned} & \sum_{\eta=n+\tau-\sigma}^{\eta-1} \lambda(n) \sum_{s=n+1}^{n-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \\ & \geq \sum_{s=N}^{\eta-1} \frac{q(n)}{r(n+\tau-\sigma)} \ln \left(\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} \frac{q(s)}{r(s+\tau-\sigma)} \right). \end{aligned} \quad (34)$$

Using (18) in (34), it follows that

$$\begin{aligned} & \sum_{\eta=n+\tau-\sigma}^{\eta-1} \lambda(n) \\ & \geq \frac{k_0(p_0 - a_0)}{p_1 p_0} \sum_{n=N_1}^{\eta-1} \frac{q(n)}{r(n+\tau-\sigma)} \ln \left(\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \right), \end{aligned}$$

or

$$\begin{aligned} & \ln \frac{y(\eta+\tau-\sigma)}{y(\eta)} \\ & \geq \frac{k_0(p_0 - a_0)}{p_1 p_0} \sum_{n=N_1}^{\eta-1} \frac{q(n)}{r(n+\tau-\sigma)} \ln \left(\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n-\tau+\sigma} \frac{q(s)}{r(s+\tau-\sigma)} \right). \end{aligned} \quad (35)$$

From (35) and (25), we have

$$\lim_{n \rightarrow \infty} \frac{y(n+\tau-\sigma)}{y(n)} = \infty. \quad (36)$$

On the otherhand, from condition (24), there exists a sequence $\{n_k\}$ of integers, $n_k \rightarrow \infty$ as $k \rightarrow \infty$, and there exists $n_k^* \in \{n_k, n_k + 1, \dots, n_k + \sigma - \tau\}$ for every k such that

$$\sum_{s=n_k}^{n_k^*} \frac{q(s)}{r(s+\tau-\sigma)} \geq \frac{C}{2} \quad \text{and} \quad \sum_{s=n_k^*}^{n_k+\sigma-\tau} \frac{q(s)}{r(s+\tau-\sigma)} \geq \frac{C}{2}. \quad (37)$$

Summing both sides of (28) from n_k to n_k^* and n_k^* to $n_k + \sigma - \tau$, we have

$$y(n_k^* + 1) - y(n_k) + \frac{k_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n_k}^{n_k^*} \frac{q(s)y(s + \tau - \sigma)}{r(s + \tau - \sigma)} \leq 0 \quad (38)$$

and

$$y(n_k + \sigma - \tau + 1) - y(n_k^*) + \frac{k_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n_k^*}^{n_k + \sigma - \tau} \frac{q(s)}{r(s + \tau - \sigma)} y(s + \tau - \sigma) \leq 0. \quad (39)$$

Using the decreasing nature of $\{y(n)\}$ and from (37), (38) and (39), we get

$$-y(n_k) + \frac{k_0(p_0 - a_0)C}{2p_1 p_0} y(n_k^* + \tau - \sigma) \leq 0$$

and

$$-y(n_k^*) + \frac{k_0(p_0 - a_0)C}{2p_1 p_0} y(n_k) \leq 0.$$

This implies eventually

$$\frac{y(n_k^* + \tau - \sigma)}{y(n_k^*)} \leq \left(\frac{2p_1 p_0}{k_0(p_0 - a_0)C} \right)^2,$$

which is a contradiction with (36). The proof is complete.

4. SOME EXAMPLES

In this section we give some examples to illustrate our results.

Example 4.1. Consider the following first order neutral delay difference equation

$$\Delta \left[\frac{1}{n} x(n) + \left(2 + \frac{1}{n} \right) x(n-2) \right] + \frac{2}{3} \left(2 + \frac{1}{n} + \frac{1}{n+3} \right) x(n-4) (2 + x^2(n-4)) = 0; \quad n \geq 4 \quad (40)$$

we have $r(n) \equiv 1$, $a(n) = \frac{1}{n}$, $q(n) = \frac{2}{3} \left(2 + \frac{1}{n} + \frac{1}{n+3} \right)$, $\tau = 2$, $\sigma = 4$, $p(n) = 2 + \frac{1}{n}$ and $a(n) = \frac{1}{n}$ $f(x(n-4)) = x(n-4)(2 + x^2(n-4))$. We can easily see that $k_0 = 2$ and $1 < 2 \leq p(n) \leq \frac{9}{4} < \infty$.

Now,

$$\sum_{n=4}^{\infty} q(n) \ln \left(\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} q(s) \right)$$

$$\begin{aligned}
&= \sum_{n=4}^{\infty} \frac{2}{3} \left(2 + \frac{1}{n} + \frac{1}{n+1} \right) \ln \left(\frac{8e}{27} \left(4 + \frac{1}{n+1} + \frac{2}{n+2} + \frac{1}{n+3} \right) \right) \\
&> \frac{2}{3} \sum_{n=4}^{\infty} \left(2 + \frac{1}{n} + \frac{1}{n+1} \right) \\
&= \infty.
\end{aligned}$$

By Theorem 3.1 every solution of (40) is oscillatory. One such solution of (40) is $x(n) = (-1)^n$.

Example 4.2. Consider the following first order neutral delay difference equation

$$\Delta \left[n \left(\frac{1}{n} x(n) + \left(2 - \frac{1}{n} \right) x(n-1) \right) \right] + \frac{2}{3} (2n-1) x(n-3) (2 + x^2(n-3)) = 0; \quad n \geq 3. \quad (41)$$

Clearly, $r(n) = n$, $a(n) = \frac{1}{n}$, $p(n) = 2 - \frac{1}{n}$, $q(n) = \frac{2}{3} (2n-1)$, $\tau = 1$, $\sigma = 3$ and

$f(x(n-3)) = x(n-3)(2 + x^2(n-3))$. We can see that $k_0 = 2$, $1 < \frac{5}{3} \leq p(n) \leq 2 < \infty$ and $a(n) = \frac{1}{3}$. Now

$$\begin{aligned}
\liminf_{n \rightarrow \infty} \sum_{s=n}^{n+\sigma-\tau} \frac{q(s)}{r(s+\tau-\sigma)} &= \frac{2}{3} \liminf_{n \rightarrow \infty} \sum_{s=n}^{n+2} \left(\frac{2s-1}{s-2} \right) \\
&= \frac{2}{3} \liminf_{n \rightarrow \infty} \sum_{s=n}^{n+2} \left(2 + \frac{3}{s-2} \right) \\
&= \frac{2}{3} \liminf_{n \rightarrow \infty} \left(6 + \frac{3}{n-2} + \frac{3}{n-1} + \frac{3}{n} \right) \\
&= 4 > 0.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{q(n)}{r(n+\tau-\sigma)} \ln \left(\frac{ek_0(p_0 - a_0)}{p_1 p_0} \sum_{s=n+1}^{n+\sigma-\tau} \frac{q(s)}{r(s+\tau-\sigma)} \right) \\
= \sum_{n=3}^{\infty} \left(\frac{q(n)}{r(n-2)} \right) \ln \left(\frac{4e}{15} \left(4 + \frac{3}{s-1} + \frac{3}{s} \right) \right)
\end{aligned}$$

$$\begin{aligned}
&> \sum_{n=3}^{\infty} \frac{q(n)}{r(n-2)} \\
&= \frac{2}{3} \sum_{n=3}^{\infty} \left(2 + \frac{3}{n-2} \right) \\
&= \infty.
\end{aligned}$$

Hence all the conditions of the Theorem 3.2 are satisfied and so every solution of (41) is oscillatory. One such solution of (41) is $x(n) = \{(-1)^n\}$.

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