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# MATHEMATICAL MODELLING OF PULSATILE FLOW OF NON-NEWTONIAN FLUID THROUGH AN ELASTIC ARTERY: EFFECTS OF ELASTICITY

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**ABSTRACT**

Analysis and understanding of various aspects of blood flow through stenosis of elastic artery is extremly important, for getting and insight into the study of arterical disease for diagonistic and clinical purposes. The cause and development of elastic arterial disease due to stenotic obstruction not only restrict the regular blood flow but also charecterizes the hardening and thickening of the arterial wall. Artery is an elastic tube whose diameter will vary with pulsating pressure, the ejection of the blood by the heart is formed the flow waves, the elastic artery of the stenosis wall is useful to find the flow of the velocity. The present paper is study the effect of non-Newtonian nature of blood flow through an elastic stenosed artery. The artery is considered as an elastic with blood is considered under power law, Herschel Bulkley model. The effect of elastic nature of stenosed arterial wall on longitudinal velocity shear stress and the flow rate is analyzed for both the models of blood under consideration.

**Key words and Phrases:** Powerlaw, Herschel-Bulkley, Pulsatileflow, Elastic artery, Elasticity, Stenosis.

## 1. INTRODUCTION

In the recent years the problem of fluid flow through the elastic artery with stenosis has received considerble attention due to its application in the cardiovascular system. The Narrowing of blood vessels commonly refered to as a stenosis, is a continuous results of elastic arterial disease. In general the stenosis of elastic artery distrub the normal blood flow, there is considerable evidence that hydrodynamic factors can play a significant role in the development and progression of this pathological condition. The flow charecterisistics of blood and the mechanical behaviour of the blood vessel wall cause may elastic arterial diseases. The behaviour of an elastic tube containing viscous fluid of pulsatile flow has been the main objective of other researches concerning the analysis of blood flow. A non-Newtonian blood flow through tapered arteries is studied by [3]. Two phase fluid model for blood flow through arteries with stenosis is studied by [8]. The Herschel-Bulkley fluid througha catherised arteries is studied by [5]. Pulsatile flow inside moderately elastic artries with effects of elasticity is studied by [7]. An application of blood flow of power law fluid through an artery with stenosis studied by [1]. Blood flow through a tapered artery with a stenosis using power law model studied by [4]. Blood flow through an atherosclerotі arterial segment using power law model studied by [2]. Pulsatile flow velocity profile in the two layer model is studied by [6].

In this paper we have developed a mathematical model of non-Newtonian blood flow through an elastic artery with constricted tube is considered. The problem is solved by the method of analytical. The effects off pulsatile flow with stenosis region of elastic artery are discussed. The aim of the present study is to analyze the effect of non-Newtonion nature of blood through a elastic mild stenosed artery. The artery is considered elastic and blood is considered under power-law model and Harschel-Bulkley model. The effect of elastic nature of stenosed arterial wall on longitudinal velocity, flowrate, wall shear stress of blood is discussed for both the model of blood for different parameters power index n and Elasticity Eh. The results are depited in graphs and comparisons are carried out for the both the model.

## 2. METHODS

### A. Formulation and Solution of the problem

A non-Newtonian blood flow through an elastic artery is considered. The vessel is considered field with blood at rest and surrounded by fluid. The axis of the vessel is taken along the axis of z. For mathematical convenience, stenosed artery considered as a long cylindrical tube so that entrance and end effect can be avoided. The following assumption are taken into account

(i) All physical properties are constants.

(ii)The flow is steay and fully developed.

(iii) The flow is axisymmetric and laminar.

(iv) The arterial wall is elastic.

The geometry of the stenosis is shown in figure.

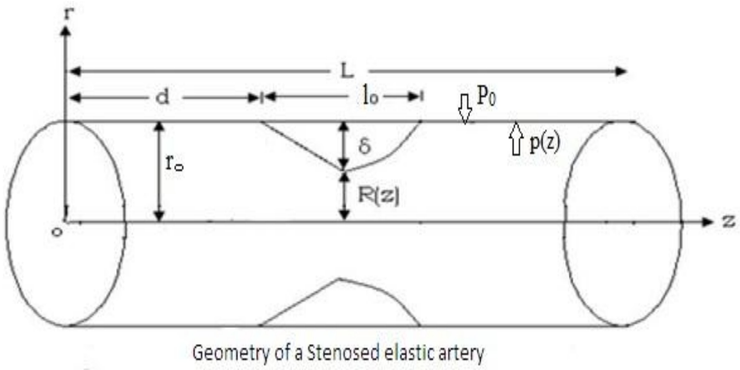


**Figure 1: Physical model of the problem**

Considering the transmural pressure difference of the vessel as described in (Mazumdar, 1992).

$$Th=r\left[p(z)-p_0\right] \tag{1}$$

where $h$ be the wall thickness of the tube, $r$ the radius of the tube, $p_0$ the exterior pressure and $p$ the interior Blood pressure, $p-p_0$ is the transmural pressure difference, $T$ is the tension per unit length and per unit thickness of the tube. According to Hooke's law the tension $T$ is defined by

$$T=E\,\frac{(r-r_0)}{r_0} \tag{2}$$

where $r=r_0$ is equilibrium position when tension $T$ is zero, $E$ is Youngs modules of the wall. The geometry of the stenosis which is assumed to be manifested in the arterial wall segment is described as:

$$\frac{R(z)}{r_0}=\begin{cases}1-\frac{\delta}{r_0}[1+\cos\cos\frac{2\pi}{l_0}(z-d-\frac{l_0}{2})];\ when\ d\le z\le d+l_0 \\ \end{cases}$$

; otherwise

Where $R(z)$ is the radius of the artery with stenosis, $R_0$ is the radius of the artery without stenosis, $\delta$ is the maximum height of the stenosis in the artery, $l_0$ is the length of the stenosis, $l$ is the length of the artery and $d$ is the location of the stenosis in the artery.

**B. The Governing equation of Motion**

Governing equation of motion of a non-Newtonian fluid in cylindrical co-ordinate system $\left(r^*, \theta^*, z^*\right)$ where $r^*$ and $z^*$ denote the radial and axial coordinates respectively and $\theta^*$ is the azimuthai angle is

$$\mu\left[\frac{1}{r^*}\frac{d}{dr^*}\left(r^*\frac{du^*}{dr^*}\right)\right] = \frac{dp^*}{dz^*} \qquad 0 \le r^* \le r_0 \tag{3}$$

where $\mu$ denotes the viscosity, $u^*$ the axial velocity, $p^*$ the pressure and $r_0$ the radius of the tube. Now, by Newton's law of viscosity

$$\tau^* = -\mu\frac{du^*}{dr^*} \tag{4}$$

From equation (3) and (4), we get

$$\frac{d}{dr^*}\left(r^*\tau^*\right) = -r^*\frac{dp^*}{dz^*}, \qquad 0 \le r^* \le r_0 \tag{5}$$

Since the shear stress distribution is independent of the type of fluid in a tube or pipe, equation (5) is valid for any kind of fluid. Integrating equation (5), we have

$$\tau^* = -\frac{1}{2}r^*\frac{dp^*}{dz^*}, \qquad 0 \le r^* \le R(z) \tag{6}$$

Here we use the condition that $\tau^*$ should be finite at $r^*=0$.

From eq. (5), $\tau^*=0$ at $r^*=0$

To find the flow field completely, $\tau^*$ must be supplemented by a constitutive equation relating shear stress and shear rate. In this problem we use non-Newtonian blood flow model namely power law model and Herschel-Bulkley model for shear stress and shear rate relation

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**C. Constitutive Equation for non-Newtonin Fluid**

The constitutive eq. of power-law model for non-Newtonian fluid flow is

$$\tau^* = \mu_\infty \left( -\frac{du^*}{dr^*} \right)^n \tag{7}$$

where $\tau^*$ is the shear stress and $\mu_\infty$ is the apparent viscosity of the fluid and constitutive equation of Herschel - Bulkley model for non-Newtonian fluid flow is,

$$\tau^* = \mu_\infty (\dot{\gamma}) + \tau_y, \dot{\gamma} = 0, \qquad \tau^* \leq \tau_y \tag{8}$$

where $\tau_y$ denotes the yield stress and $\mu_\infty$ is the apparent viscosity of the fluid. As velocity gradient $\dot{\gamma}$ in the region where shear stresss is less than yield stress is vanishing, so, plug flow exists whenever $\tau^* \leq \tau_y$. The radius of the plug core region is, say, $r_p$ for $r^* \leq r_p, \tau^* \leq \tau_y$ . Thus, for the core region $\dot{\gamma}=0$.

$$i.e. \qquad \frac{du^*}{dr^*} = 0. \tag{9}$$

**D. Boundary Condition**

Here we consider no slip boundary condition at the wall $r^* = R(z)$ of the tube i.e. the component of fluid velocity vanish at all points of the surfaces of the stationary body with which the fluid may be in contact i.e. $u^* = 0$ at $r^* = R(z)$.

## 3. METHOD OF SOLUTION

Introducing following non-dimensional variables,

$$r = \frac{r^*}{r_0}, u = \frac{u^*}{u_0}, \tau = \frac{\tau^*}{\tau_0} \tag{10}$$

where $$u_0 = \frac{r_0^2}{2\mu_\infty} \frac{dp^*}{dz^*} \tag{11}$$

and $$\tau_0 = \frac{\mu_\infty}{r_0} u_0 \tag{12}$$

are characteristic values of velocity and shear stress respectively.

Using the relation (1) in (2) we have,

$$r = \frac{r_0}{1 - r_0 (p(z) - p_0)/Eh} \tag{13}$$

Using (8) and (9) the non-dimensional form of constitutive equation be

$$t = r, \quad 0 \text{£} r \text{£} R(z)/r_0 \tag{14}$$

Now, from the eq. (11) and (12) the constitutive equation of flow of a Newtonian fluid in elastic tube is

$$t = \frac{r_0}{1 - r_0 (p(z) - p_0)/Eh} \tag{15}$$

The constitutive eq. of power law model (5) in non-dimensional form becomes

$$\left(-\frac{du}{dr}\right) = \left(\frac{m_{¥}}{t_0}\right)^{\frac{n-1}{n}} \times (t)^{\frac{1}{n}} \tag{16}$$

On incorporating dimensionless variables in the Herschel Bulkley eq. (6), we get

**Error!** (17)

**Error!**

Where, **Error!** (18)

The boundary condition in non-dimensional form is reduced to

$$u = 0 \quad at \quad r = R(z)/r_0. \tag{19}$$

**A. Calculation of velocity for power law model**

$$\frac{du}{dz} = \frac{du}{dr} \times \frac{dr}{dz} \tag{20}$$

Now using eq. (11), (12) and (13), the above expression gives

$$u = \frac{A_1}{Eh} \int (Eh)^{\frac{1}{n}+2} \left(\frac{1}{\frac{Eh}{r_0} - \left(p(z) - p_0\right)}\right)^{\frac{1}{n}+2} \frac{dp(z)}{dz} dz,$$

Where

$$A_1 = -\left(\frac{m_{¥}}{t_0}\right)^{\frac{n-1}{n}} \tag{21}$$

$$\frac{Eh}{r_0} - \left[p\,(z) - p_0\right] = z => -p(z) = dz \qquad (22)$$

$$\text{Then } u = -A_1 (Eh)^{\frac{1}{n}+1} \left( \frac{(z)^{-\frac{1}{n}-1}}{-\frac{1}{n}-1} \right) + C$$

using the equation (11), (19), and no slip condition $u=0$ at $r = \frac{R(z)}{r_0}$

$$u = \frac{A_1 n}{n+1} \left( -\frac{R(z)}{r_0} + (r)^{\frac{1}{n}+1} \right) \qquad (23)$$

**B. Calculation of volume flow rate for power law model**

$$Q = 4 \int_0^{\frac{R(z)}{r_0}} \frac{du}{dr} r^2 dr$$

$$= 4 \int_0^{\frac{R(z)}{r_0}} \left(\frac{\mu_\infty}{\tau_0}\right)^{\frac{n-1}{n}} (\tau)^{\frac{1}{n}} r^2 dr$$

Using □=$r$ for 0□$r$□1, we get

$$\mathbf{Q} = \frac{4n}{3n+1} \left( \frac{m_¥}{t_0} \right)^{\frac{n-1}{n}} \left[ \frac{R(z)}{r_0} \right]^{\frac{3n+1}{n}} \qquad (24)$$

**C. Calculation of wall shear stress for power law model**

The wall shear stress for power law model in the elastic stenosed artery is defined by

$$t_w = -[m \frac{du}{dr}]_{r=R(z)}$$

$$t_w = (-1)^{1+\frac{1}{n}} (m) \left( \frac{A_1 n}{n+1} \right) \times \left( \frac{1}{n} + 1 \right) \left( 1 - \frac{R(z)}{r_0} \right)^{\frac{1}{n}}$$

**D. Calculation of velocity for Hershel bulkley model**

We have from eq. (11) and (15)

**Error!** (25)

Where **Error!**

Put $\frac{Eh}{r_0}$-($p$ $(z)$-$p_0$=z) **Error!**, so we get

$$du=\left(\frac{a_1}{Eh}\right)\left(\frac{a_3}{a_2}-1\right)^{\frac{1}{n}}\left(\frac{a_4}{(a_2)^2}\right)dz$$

where $a_3=\frac{E_h}{t}, a_4=(Eh)^2$ integrating (22) w.r.to □ and then using no slip condition to find constant of integration, we get

$$u=-\frac{A_1 n}{(n+1)}(t)^{\frac{1}{n}}\left(-1+\frac{r}{t}\right)^{\frac{1}{n}}(t-r)-\left(-1+\frac{R(z)}{r_0 t}\right)^{\frac{1}{n}}\left(t-\frac{R(z)}{r_0}\right)$$

**E. Calculation of Volume Flow Rate For Hershel Bulkley Model**

The non-dimensional flow rate can be computed as

$$Q=8\int_0^{\frac{R(z)}{r_0}} urdr \quad =\left[8u\frac{r^2}{2}\right]_0^{\frac{R(z)}{r_0}}-8\int_0^{\frac{R(z)}{r_0}}\frac{du}{dr}\frac{r^2}{2}dr$$

$$\mathbf{Q}=4\left(\frac{\mu_\infty}{\tau_0}\right)^{\frac{n-1}{n}}\left[\left[\left(\frac{n}{n+1}\right)\left(\frac{R(z)}{r_0}\right)^2\left(\frac{R(z)}{r_0}-\hat{\tau}\right)^{\frac{1}{n}}-\left(\frac{2n}{2n+1}\right)\left(\frac{R(z)}{r_0}\right)\left(\frac{R(z)}{r_0}-\hat{\tau}\right)^{\frac{1}{n}+2}+\left(\frac{2n}{3n+1}\right)\left(\frac{R(z)}{r_0}-\hat{\tau}\right)^{\frac{1}{n}+3}-\left(\frac{2n}{3n+1}\right)\left(-\hat{\tau}\right)^{\frac{1}{n}+3}\right]\right]$$

**F. Calculation of wall shear stress for Hershel bulkley model**

The wall shear stress for Herschel Bulkley model in the elastic stenosed artery is defined by $t_w=-[m\,\frac{du}{dr}]_{r=R(z)}$

$$t_w=\frac{A_1 n}{n+1}m\,t^{\frac{1}{n}}\left[\left(-1+\frac{R(z)}{r_0 t}\right)^{\frac{1}{n}}\left(-\frac{R(z)}{r_0 t}+\frac{t-r}{n}\left(-1+\frac{R(z)}{r_0 t}\right)^{\frac{1}{n}-1}\right)\right]$$

## 4. CONCLUSION

The artery is considered as an elastic with stenosis, blood is considered under power-law model and Herschel-Bulkley model. The comparison in flow profiles for power-law model of blood with the Herschel-bulkley model is demonstrated in the figures. In figure 2, in the elastic stenosed artery the velocity profiles for power law model is advanced compare to the Herschel-Bulkley model for fixed value of power index n and other parameter. Figure 2 shows that with the increase of power index the velocity of fluid decreases for both the fluids. In Fig.3,we observe that in power law model, the linear transmula pressure the velocity of the blood increases with increase of elasticity at different locations of stenosed region R(z). In Fig.4, we observe that in Herschel Bulkley model, the linear transmula pressure the velocity of the blood decreases with increase of elasticity at different locations of stenosed region R(z). In fig.5 for the powerlaw model , the volume flow rate decreases with increase of power index n at different location of stenosed region R(z). In Fig.6 for the Herschel-Bulkley model, the volumetric flow rate increases with increase of power index n at different location of stenosed region R(z). Also we see that in the elastic stenosed artery, the volume flow rate of Herschel-Bulkley model is advanced when compare to the powerlaw model for different values of power index n. In Figure 7, we see that in the elastic stenosed artery the wall shear stress for power law model is advanced compare to the Herschel-Bulkley model for fixed value of power index and other parameter. Figure 7, shows that the wall shear stress decreases as the increases of power n, at different location of stenosed region R(z) for both the model. The theoreical results provides a scope of many results on physiological properties of blood flow through elastic arteries with stenosis by using the power law model and Herschel Bulkley model. Also, it is notice that from the physiological properties, the length and thickness of the stenosed elastic arteries influence the flow charecteristic of the blood flow.

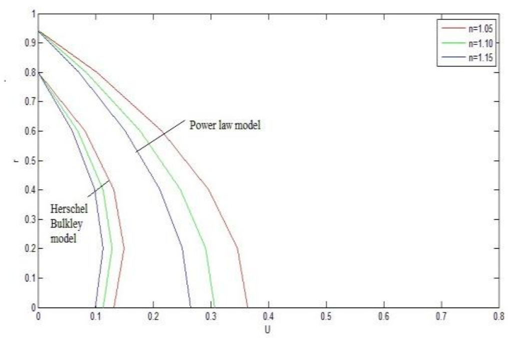


**Figure 2: Velocity profiles for PowerLaw model and HerschelBulkley model for different value of power index(n) at Yield stress=0.2**

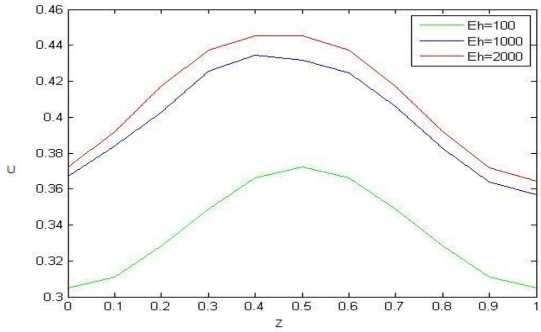


**Figure 3: Effect of elasticity on velocity profile for power law model at different location along stenosed region R(z) n=0.75**

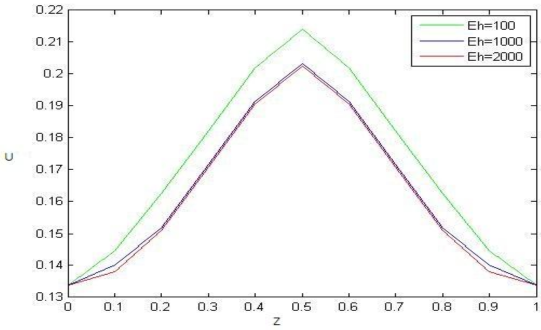


**Figure 4: Effect of elasticity on velocity for HerschelBulkley model at different location along stenosed region R(z) (At n=0.75,Yield stress=0.2)**

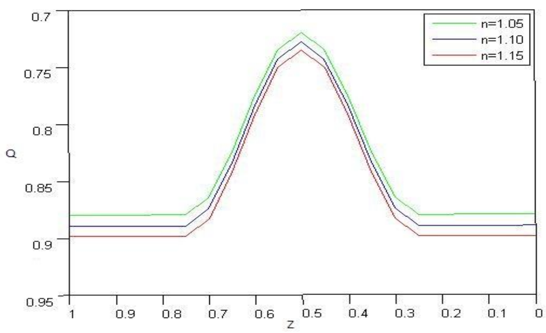


**Figure 5: Variation of volume flow rate with stenosed region *R*(*z*) for different values of n for power law model with yield stress =0.2, □=0.2,*Eh*=100**

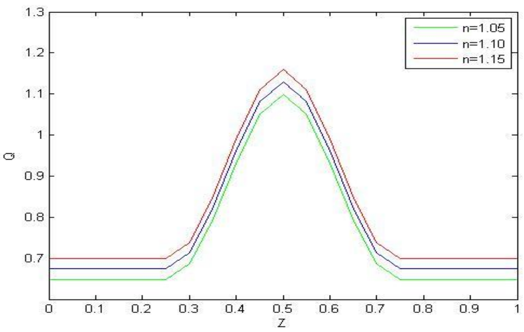


**Figure 6: Volume flow rate versus power index (n) for Herschel Bulkley model (At Eh=100, Yield stress 0.2)**

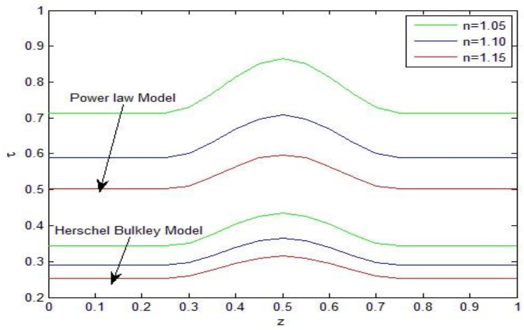


**Figure 7: Variation of wall with stenosed region R(z) shear stress radially for different values of power index n for Power law model and Herschel bulkley model at yield stress = 0.2**

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