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STABILITY OF DGC'S QUADRATIC FUNCTIONAL EQUATION IN QUASI-BANACH ALGEBRAS: DIRECT AND FIXED POINT METHODS

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ABSTRACT

In this paper, we obtain the general solution and the generalized Ulam-Hyers stability of Degen-Graves-Cayley -Eight Squares (DGC's) quadratic functional equation of the form

$$\begin{aligned}\left(\sum_{i=1}^{8} f(x_i)\right)\left(\sum_{i=1}^{8} f(y_i)\right) = {} & f\left(x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8\right)\\
& + f\left(x_2y_1 + x_1y_2 + x_4y_3 - x_3y_4 + x_6y_5 - x_5y_6 + x_8y_7 - x_7y_8\right)\\
& + f\left(x_3y_1 - x_4y_2 + x_1y_3 + x_2y_4 + x_7y_5 + x_8y_6 - x_5y_7 - x_6y_8\right)\\
& + f\left(x_4y_1 + x_3y_2 - x_2y_3 + x_1y_4 + x_8y_5 - x_7y_6 + x_6y_7 - x_5y_8\right)\\
& + f\left(x_5y_1 - x_6y_2 - x_7y_3 - x_8y_4 + x_1y_5 + x_2y_6 + x_3y_7 + x_4y_8\right)\\
& + f\left(x_6y_1 + x_5y_2 - x_8y_3 + x_7y_4 - x_2y_5 + x_1y_6 - x_4y_7 + x_3y_8\right)\\
& + f\left(x_7y_1 + x_8y_2 + x_5y_3 - x_6y_4 - x_3y_5 + x_4y_6 + x_1y_7 - x_2y_8\right)\\
& + f\left(x_8y_1 - x_7y_2 + x_6y_3 + x_5y_4 - x_4y_5 - x_3y_6 + x_2y_7 + x_1y_8\right)\end{aligned}$$

in quasi-Banach algebras using direct and fixed point methods.

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1. Introduction

The stability problem of a functional equation was first posed by Ulam [41] concerning the stability of group homomorphism which was answered by Hyers [21] for Banach spaces. Hyers' theorem was generalized by T. Aoki [2] for additive mappings and by Th.M. Rassias [36] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [36] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability of functional equations. A generalization of the Th.M. Rassias theorem was obtained by P. Gavruta [18] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassia's approach. In 1982, J.M. Rassias [31] followed the innovative approach of the Th.M. Rassias theorem [36] in which he replaced the factor $\| x\|^p + \| y\|^p$ by $\| x\|^p\| y\|^q$ for $p,q \in \mathrm{P}$ with $p+q \neq 1$.

In 1994, the above stability results were further extended by P. P. Găvruta [18] who considered a more general control function in the real variables x, y for the unbounded Cauchy difference in the spirit of Th.M. Rassias' stability approach. In 2008, a special case of Găvruta theorem for the unbounded Cauchy difference was obtained in [37] by considereing the summation of both the sum and product of two $p-$ norms.

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

and therefore the equation (1) is called quadratic functional equation.

In 2014, M. Arunkumar and S. Karthikeyan [4] introduced and investigated the Ulam-Hyers stability of Brahmagupta quadratic functional equations

$$\left(f(x_1)+nf(x_2)\right)\left(f(x_3)+nf(x_4)\right)= f\left(x_1x_3 \pm nx_2x_4\right)+nf\left(x_1x_4 \mp x_2x_3\right) \tag{1.2}$$

on non-Archimedean Banach algebras using direct and fixed point methods.

In 2015, John M. Rassias, M. Arunkumar and S. Karthikeyan [33] introduced and proved the generalized Ulam-Hyers stability of Lagrange's quadratic functional equation

$$\left(\sum_{i=1}^{n} f(x_i)\right)\left(\sum_{i=1}^{n} f(y_i)\right) = f\left(\sum_{i=1}^{n} x_i y_i\right) + \sum_{1\le i<j\le n} f\left(x_i y_j - x_j y_i\right) \tag{1.3}$$

where n is a positive integer in Lie C^*-algebras using direct and fixed point methods.

Recently, John M. Rassias, M. Arunkumar and S. Karthikeyan [34] obtain the general solution and the generalized Ulam-Hyers stability of Euler's quadratic functional equation of the form

$$\begin{aligned}\left(\sum_{i=1}^{4} f(x_i)\right)\left(\sum_{i=1}^{4} f(y_i)\right) &= f\left(x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4\right) + f\left(x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3\right) \\ &\quad + f\left(x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2\right) + f\left(x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1\right)\end{aligned} \tag{1.4}$$

in JC^*-algebras using direct and fixed point methods. An application of this functional equation is also studied.

In this paper, we obtain the general solution and the generalized Ulam-Hyers stability of Degen-Graves-Cayley -Eight Squares (DGC's) quadratic functional equation of the form

$$\begin{aligned}\left(\sum_{i=1}^{8} f(x_i)\right)\left(\sum_{i=1}^{8} f(y_i)\right) &= f\left(x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8\right) \\ &\quad + f\left(x_2y_1 + x_1y_2 + x_4y_3 - x_3y_4 + x_6y_5 - x_5y_6 + x_8y_7 - x_7y_8\right) \\ &\quad + f\left(x_3y_1 - x_4y_2 + x_1y_3 + x_2y_4 + x_7y_5 + x_8y_6 - x_5y_7 - x_6y_8\right) \\ &\quad + f\left(x_4y_1 + x_3y_2 - x_2y_3 + x_1y_4 + x_8y_5 - x_7y_6 + x_6y_7 - x_5y_8\right) \\ &\quad + f\left(x_5y_1 - x_6y_2 - x_7y_3 - x_8y_4 + x_1y_5 + x_2y_6 + x_3y_7 + x_4y_8\right) \\ &\quad + f\left(x_6y_1 + x_5y_2 - x_8y_3 + x_7y_4 - x_2y_5 + x_1y_6 - x_4y_7 + x_3y_8\right) \\ &\quad + f\left(x_7y_1 + x_8y_2 + x_5y_3 - x_6y_4 - x_3y_5 + x_4y_6 + x_1y_7 - x_2y_8\right) \\ &\quad + f\left(x_8y_1 - x_7y_2 + x_6y_3 + x_5y_4 - x_4y_5 - x_3y_6 + x_2y_7 + x_1y_8\right)\end{aligned} \tag{1.5}$$

in quasi-Banach algebras using direct and fixed point methods.

2. Preliminaries

We now give some basic definitions concerning quasi-Banach space and some preliminary results.

Definition 2.1

Let X *be a real linear space. A quasi-norm is a real-valued function on* X *satisfying the following:*

(1) $\|x\| > 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|.\|x\|$ for all $\lambda \in \mathrm{P}$ and for all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|x + y\| \geq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(A, \|.\|)$ is called a *quasi-normed space* if $\|.\|$ is a *quasi norm* on X . The smallest possible K is called the *modulus of concavity* of $\|.\|$. A *quasi-Banach space* is a complete *quasi-normed space*.

Definition 2.2

A quasi-norm $\|.\|$ *is called a* p *-norm* $(0 < p \leq 1)$ *if* $\|x + y\|^p \geq \|x\|^p + \|y\|^p$ *for all* $x, y \in X$ *. In this case, a quasi-Banach space is called a* p *-Banach space.*

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X . By the Aoki-Rolewicz theorem [38] (see also [8]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

Definition 2.3

Let $(A, \|.\|)$ *be a quasi-normed space. The quasi-normed space* $(A, \|.\|)$ *is called a quasi-normed algebra if* A *is an algebra and there is a constant* $C > 0$ *such that* $\|xy\| \leq C\|x\|\|y\|$ *for all* $x, y \in A$.

Definition 2.4

A quasi-Banach algebra is a complete quasi-normed algebra. If the quasi-norm $\|.\|$ *is a* p *-norm, then the quasi-Banach algebra is called a* p *-Banach algebra.*

Bourgin [9] proved the stability of ring homomorphisms in two unital Banach algebras. Badora [5] gave a generalization of the Bourgin's result. The stability result concerning derivations on operator algebras was first obtained by $\hat{S}emrl$ [39]. In [6], Badora proved the stability of functional equation $f(xy) = xf(y) + f(x)y$, where f is a mapping on normed algebra A with unit.

Definition 2.5

Let A, B *be two algebras. A mapping* $f : \mathrm{A} \to \mathrm{B}$ *is called a quadratic homomorphism if* f *is a quadratic mapping satisfying* $f(xy) = f(x) f(y)$ *for all* $x, y \in \mathrm{A}$*. For instance, let* A *be commutative. Then the mapping* $f : \mathrm{A} \to \mathrm{A}$*, defined* *by* $f(x) = x^2 (x \in \mathrm{A})$, *is a quadratic homomorphism.*

Definition 2.6

A mapping $f : \mathrm{A} \to \mathrm{A}$ *is called a quadratic derivation if* f *is a quadratic mapping satisfying* $f(xy) = x^2 f(y) + f(x) y^2$ *for all* $x, y \in \mathrm{A}$.

We note that quadratic derivations and ring derivations are different.

3 General Solution of the Functional Equation (1.5)

In this section, the authors investigate the general solution of the functional equation (1.5). Through out this section, let us consider X and Y be real vector spaces.

Theorem 3.1

If the mapping $f : X \to Y$ *satisfies the functional equation* (1.5) *for all* $x_1, y_1, x_2, y_2, \ldots, x_8, y_8 \in X$ *then* $f : X \to Y$ *satisfying the functional equation* (1.1) *for all* $x, y \in X$.

Proof. Setting $x_3 = y_3 = \ldots = x_8 = y_8 = 0$ in (1.5), we get

$$\left(f(x_1) + f(x_2)\right)\left(f(y_1) + f(y_2)\right) = f(x_1 y_1 - x_2 y_2) + f(x_2 y_1 + x_1 y_2) \tag{3.1}$$

for all $x_1, y_1, x_2, y_2 \in X$. Replacing (x_1, y_1, x_2, y_2) by $(\sqrt{x}, 0, \sqrt{x}, 0)$ in (3.1), we obtain

$$\left(f\left(\sqrt{x}\right)\right)^2 = f(x) \tag{3.2}$$

for all $x \in X$. Setting (x_1, y_1, x_2, y_2) by $(\sqrt{x}, 0, \sqrt{y}, 0)$ in (3.1), we get

$$f\left(\sqrt{x}\right) f\left(\sqrt{y}\right) = f\left(\sqrt{x}\sqrt{y}\right) \tag{3.3}$$

for all $x, y \in X$. Replacing (x_1, y_1, x_2, y_2) by $(\sqrt{x}, \sqrt{x}, \sqrt{x}, \sqrt{x})$ in (3.1) and using (3.2) and using (3.3), we arrive

$$f(2x) = 2^2 f(x) \tag{3.4}$$

for all $x \in X$. Letting (x_1, y_1, x_2, y_2) by $(x, 0, x, 0)$ in (3.1), we obtain

$$\left(f(x)\right)^2 = f(x^2)$$

for all $x \in X$. It can be rewritten as

$$f(x) = \sqrt{f(x^2)} \tag{3.5}$$

for all $x \in X$. Replacing x by $-x$ in (3.5), we get $f(-x) = f(x)$ is an even function. Setting $((x_1, y_1, x_2, y_2)$ by $(\sqrt{x}, \sqrt{y}, \sqrt{x}, -\sqrt{y})$ in (1) and using (3.2), (3.4), we get

$$f(x) + f(y) + 2f\left(\sqrt{x}\right)f\left(\sqrt{y}\right) = f(x+y) \tag{3.6}$$

for all $x, y \in X$. Letting (x_1, y_1, x_2, y_2) by $(\sqrt{x}, \sqrt{y}, \sqrt{x}, \sqrt{y})$ in (3.1) and using (3.2), (3.3), (3.4), we obtain

$$f(x) + f(y) - 2f\left(\sqrt{x}\right)f\left(\sqrt{y}\right) = f(x-y) \tag{3.7}$$

for all $x, y \in X$. Adding (3.6) and (3.7), we derive (1.1) for all $x, y \in X$.

Hence the proof is complete.

Hereafter throughout this paper, assume that A is a quasi-normed algebra with quasi-norm $\|\cdot\|_{\mathrm{A}}$ and that B is a p-Banach algebra with p-norm $\|\cdot\|_{\mathrm{B}}$. For convenience, we define a mapping $f : \mathrm{A} \to \mathrm{B}$ by

$$\begin{aligned}
F(x_1, y_1, x_2, y_2, \ldots, x_8, y_8) = {} & \left(\sum_{i=1}^{8} f(x_i)\right)\left(\sum_{i=1}^{8} f(y_i)\right) \\
& - f\left(x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8\right) \\
& - f\left(x_2y_1 + x_1y_2 + x_4y_3 - x_3y_4 + x_6y_5 - x_5y_6 + x_8y_7 - x_7y_8\right) \\
& \ldots \qquad\qquad \ldots \qquad\qquad \ldots \\
& - f\left(x_8y_1 - x_7y_2 + x_6y_3 + x_5y_4 - x_4y_5 - x_3y_6 + x_2y_7 + x_1y_8\right)
\end{aligned}$$

for all $x_1, y_1, \ldots, x_8, y_8 \in \mathrm{A}$.

4. Stability of Quadratic Homomorphisms and Derivations of (1.5): A Direct Method

In this section, the authors present the generalized Ulam-Hyers stability of the functional equation (1.5).

Definition 4.1

A X*-linear mapping* $\mathrm{H} : \mathrm{A} \to \mathrm{B}$ *is called a quadratic homomorphism in quasi-banach algebras if*

$$\mathrm{H}(xy) = \mathrm{H}(x)\mathrm{H}(y) \tag{4.1}$$

for all $x, y \in \mathrm{A}$.

Theorem 4.2

Let $j \in \{-1,1\}$. *Let* $\alpha : \mathrm{A}^{16} \to [0,\infty)$ *be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(2^{kj} x_1, 2^{kj} y_1, \ldots, 2^{kj} x_8, 2^{kj} y_8\right)}{2^{2kj}} \textit{converges to } □ \tag{4.2}$$

and
$$\lim_{k\to\infty} \frac{\alpha\left(2^{kj} x_1, 2^{kj} y_1, \ldots, 2^{kj} x_8, 2^{kj} y_8\right)}{2^{2kj}} < \infty, \tag{4.3}$$

and $f : \mathrm{A} \to \mathrm{B}$ be a mapping satisfies the inequality

$$\left\|\mathrm{F}(x_1, y_1, \ldots, x_8, y_8)\right\|_B \le \alpha\left(x_1, y_1, \ldots, x_8, y_8\right) \tag{4.4}$$

for all $x_1, y_1, \ldots, x_8, y_8 \in \mathrm{A}$ and

$$\| f(xy) - f(x) f(y)\|_B \le \alpha(x, y, 0, 0, \ldots, 0, 0) \tag{4.5}$$

for all $x, y \in \mathrm{A}$. Then there exists a unique quadratic homomorphism $\mathrm{H} : \mathrm{A} \to \mathrm{B}$ which satisfies (1.5) and

$$\| f(x) - \mathrm{H}(x)\|_B \le \frac{1}{28} \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\alpha\left(\left(\sqrt{2}\right)^{ij} \sqrt{x}, \ldots, \left(\sqrt{2}\right)^{ij} \sqrt{x}\right)}{2^{2ij}} \tag{4.6}$$

for all $x \in \mathrm{A}$. The mapping $\mathrm{H}(x)$ is defined by

$$\mathrm{H}(x) = \lim_{k\to\infty} \frac{f(2^{kj} x)}{2^{2kj}} \tag{4.7}$$

for all $x \in \mathrm{A}$.

Proof. Assume $j = 1$.

Replacing $(x_1, y_1, \ldots, x_8, y_8)$ by $\left(\sqrt{x}, \sqrt{x}, \ldots, \sqrt{x}, \sqrt{x}\right)$ and dividing by 28 in (4.4), we get

$$\left\|\frac{f(2x)}{2^2} - f(x)\right\|_B \le \frac{1}{28} \alpha\left(\sqrt{x}, \sqrt{x}, \ldots, \sqrt{x}, \sqrt{x}\right) \tag{4.8}$$

for all $x \in \mathrm{A}$. Replacing x by $2x$ in (4.8) and divided by 2^2, we have

$$\left\|\frac{f(2^2x)}{2^4}-\frac{f(2x)}{2^2}\right\|_B \le \frac{1}{2^2.28}\alpha\left(\sqrt{2x},\sqrt{2x},\ldots,\sqrt{2x},\sqrt{2x}\right) \tag{4.9}$$

for all $x\in \mathrm{A}$. Combining (4.8) and (4.9), we obtain

$$\left\|\frac{f(2^2x)}{2^4}-f(x)\right\|_B \le \frac{1}{28}\left(\alpha\left(\sqrt{x},\sqrt{x},\ldots,\sqrt{x},\sqrt{x}\right)+\frac{1}{2^2}\alpha\left(\sqrt{2x},\sqrt{2x},\ldots,\sqrt{2x},\sqrt{2x}\right)\right) \tag{4.10}$$

for all $x\in \mathrm{A}$. Using induction on a positive integer k , we obtain that

$$\left\|\frac{f(2^kx)}{2^{2k}}-f(x)\right\|_B \le \frac{1}{2^2.7}\sum_{i=0}^{k-1}\frac{1}{2^{2i}}\alpha\left(\sqrt{2^ix},\sqrt{2^ix},\ldots,\sqrt{2^ix},\sqrt{2^ix}\right) \tag{4.11}$$

$$\le \frac{1}{2^2.7}\sum_{i=0}^{\infty}\frac{1}{2^{2i}}\alpha\left(\sqrt{2^ix},\sqrt{2^ix},\ldots,\sqrt{2^ix},\sqrt{2^ix}\right)$$

for all $x\in \mathrm{A}$. In order to prove the convergence of the sequence $\left\{\frac{f(2^kx)}{2^{2k}}\right\}$, replace x by 2^mx and dividing by 2^{2m} in (4.11), for any $m,k>0$, we arrive

$$\left\|\frac{f(2^k2^mx)}{2^{2k+2m}}-\frac{f(2^mx)}{2^{2m}}\right\|_B = \frac{1}{2^{2m}}\left\|\frac{f(2^k2^mx)}{2^{2k}}-f(2^mx)\right\|_B$$

$$\le \frac{1}{2^2.7}\sum_{i=0}^{k-1}\frac{1}{2^{2(i+m)}}\alpha\left(\sqrt{2^{i+m}x},\sqrt{2^{i+m}x},\ldots,\sqrt{2^{i+m}x},\sqrt{2^{i+m}x}\right)$$

$$\le \frac{1}{2^2.7}\sum_{i=0}^{\infty}\frac{\alpha\left(\sqrt{2^{i+m}x},\sqrt{2^{i+m}x},\ldots,\sqrt{2^{i+m}x},\sqrt{2^{i+m}x}\right)}{2^{2(i+m)}} \tag{4.12}$$

for all $x\in \mathrm{A}$. Since the right hand side of the inequality (4.12) tends to 0 as $m\to\infty$, the sequence $\left\{\frac{f(2^kx)}{2^{2k}}\right\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $\mathrm{H}:\mathrm{A}\to\mathrm{B}$ such that

$$\mathrm{H}(x)=\lim_{k\to\infty}\frac{f(2^kx)}{2^{2k}},\ \forall\ x\in\mathrm{A}.$$

Letting $k\to\infty$ in (4.11), we see that (4.6) holds for all $x\in\mathrm{A}$. Now, we need to prove H satisfies (1.5), replacing $(x_1,y_1,\ldots,x_8,y_8)$ by $(2^kx_1,2^ky_1,\ldots,2^kx_8,2^ky_8)$ and dividing by 2^{2k} in (4.4), we arrive

$$\frac{1}{2^{2k}}\left\|F\left(2^k x_1, 2^k y_1, \ldots, 2^k x_8, 2^k y_8\right)\right\|_B \le \frac{1}{2^{2k}}\alpha\left(2^k x_1, 2^k y_1, \ldots, 2^k x_8, 2^k y_8\right)$$

for all $x_1, y_1, \ldots, x_8, y_8 \in \mathrm{A}$. Letting $k \to \infty$ in the above inequalities, we arrive

$$\left\|\mathrm{H}\left(x_1, y_1, \ldots, x_8, y_8\right)\right\|_B = 0.$$

Hence H satisfies (1.5) for all $x_1, y_1, \ldots, x_8, y_8 \in \mathrm{A}$. This shows that H is quadratic. Also

$$\left\|\mathrm{H}(x_1 y_1) - \mathrm{H}(x_1)\,\mathrm{H}(y_1)\right\|_B = \lim_{k\to\infty}\frac{1}{2^{4k}}\left\|f\left(2^{2k} x_1 y_1\right) - f\left(2^k x_1\right) f\left(2^k y_1\right)\right\|_B$$
$$\le \lim_{k\to\infty}\frac{1}{2^{4k}}\alpha\left(2^k x_1, 2^k y_1, 0, 0, \ldots, 0, 0\right) = 0$$

for all $x_1, y_1 \in \mathrm{A}$. Therefore, H is a quadratic homomorphism. In order to prove H is unique, let $\mathrm{H}'(x)$ be another quadratic homomorphism satisfying (4.6) and (1.5). Then

$$\left\|\mathrm{H}(x) - \mathrm{H}'(x)\right\|_B = \frac{1}{2^{2k}}\left\|\mathrm{H}(2^k x) - \mathrm{H}'(2^k x)\right\|_B$$

$$\le \frac{1}{2^{2k}}\left\{\left\|\mathrm{H}(2^k x) - f(2^k x)\right\|_B + \left\|f(2^k x) - \mathrm{H}'(2^k x)\right\|_B\right\}$$

$$\le \frac{2}{2^2.7}\sum_{i=0}^{\infty}\frac{1}{2^{2(i+k)}}\alpha\left(\sqrt{2^{i+k}x}, \sqrt{2^{i+k}x}, \ldots, \sqrt{2^{i+k}x}, \sqrt{2^{i+k}x}\right)$$

$$\to 0 \ as\, k \to \infty$$

for all $x \in \mathrm{A}$. Hence H is unique. Thus the mapping $\mathrm{H} : \mathrm{A} \to \mathrm{B}$ is a unique quadratic homomorphism mapping satisfying (4.6).

For $j = -1$, we can prove the similar stability result. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 4.2 concerning the stability of (1.5).

Corollary 4.3 *Let* λ *and* s *be nonnegative real numbers. If a function* $f : \mathrm{A} \to \mathrm{B}$ *satisfies the inequality*

$$\left\|\mathrm{F}(x_1,y_1,\ldots,x_8,y_8)\right\|_B \le \begin{cases} \lambda, \\ \lambda\sum_{i=1}^{8}\left\{\|x_i\|_A^s+\|y_i\|_A^s\right\}, \\ \lambda\left\{\prod_{i=1}^{8}\|x_i\|_A^s\|y_i\|_A^s\right\}, \\ \lambda\left\{\prod_{i=1}^{8}\|x_i\|_A^s\|y_i\|_A^s+\sum_{i=1}^{8}\left(\|x_i\|_A^{16s}+\|y_i\|_A^{16s}\right)\right\} \end{cases} \tag{4.13}$$

for all $x_1,y_1,\ldots,x_8,y_8\in\mathrm{A}$ and

$$\left\|f\left(x,y\right)-f(x)f(y)\right\|_B \le \begin{cases} \lambda, \\ \lambda\left\{\|x\|_A^s+\|y\|_A^s\right\}, \\ \lambda\left\{\|x\|_A^s\|y\|_A^s\right\}, \\ \lambda\left\{\|x\|_A^s\|y\|_A^s+\left(\|x\|_A^{2s}+\|y\|_A^{2s}\right)\right\} \end{cases} \tag{4.14}$$

for all $x,y\in\mathrm{A}$. Then there exists a unique quadratic homomorphism $\mathrm{H}:\mathrm{A}\to\mathrm{B}$ such that

$$\left\|f(x)-\mathrm{H}(x)\right\|_B \le \begin{cases} \dfrac{\lambda}{21}, \\ \dfrac{16\lambda\|x\|_A^{s/2}}{7|2^2-2^{\frac{s}{2}}|}, & s\ne 4; \\ \dfrac{\lambda\|x\|_A^{8s}}{7|2^2-2^{8s}|}, & s\ne \dfrac{1}{4}; \\ \dfrac{17\lambda\|x\|_A^{8s}}{7|2^2-2^{8s}|}, & s\ne \dfrac{1}{4} \end{cases} \tag{4.15}$$

for all $x\in\mathrm{A}$.

Definition 4.4

A X*-linear mapping* $\mathrm{D}:\mathrm{A}\to\mathrm{A}$ *is called a quadratic derivation in quasi-banach algebras if*

$$\mathrm{D}\left(xy\right)=\mathrm{D}\left(x\right)y^2+x^2\,\mathrm{D}\left(y\right) \tag{4.16}$$

for all $x,y\in\mathrm{A}$.

Theorem 4.5

Let $j\in\{-1,1\}$. *Let* $\alpha:\mathrm{A}^{16}\to[0,\infty)$ *be a function such that*

$$\sum_{k=0}^{\infty}\frac{\alpha\left(2^{kj}x_1,2^{kj}y_1,\dots,2^{kj}x_8,2^{kj}y_8\right)}{2^{2kj}}\ \textit{converges to}\ \square\ \ \textit{and} \tag{4.17}$$

$$\lim_{k\to\infty}\frac{\alpha\left(2^{kj}x_1,2^{kj}y_1,\dots,2^{kj}x_8,2^{kj}y_8\right)}{2^{2kj}}<\infty, \tag{4.18}$$

and $f:\mathrm{A}\to\mathrm{A}$ be a function satisfies the inequality

$$\left\|\mathrm{F}(x_1,y_1,\dots,x_8,y_8)\right\|_A\le\alpha\left(x_1,y_1,\dots,x_8,y_8\right) \tag{4.19}$$

for all $x_1,y_1,\dots,x_8,y_8\in\mathrm{A}$ and

$$\left\|f(xy)-x^2f(y)-f(x)y^2\right\|_A\le\alpha(x,y,0,0,\dots,0,0) \tag{4.20}$$

for all $x,y\in\mathrm{A}$. Then there exists a unique quadratic derivation $\mathrm{D}:\mathrm{A}\to\mathrm{A}$ which satisfies (1.5) and

$$\|f(x)-\mathrm{H}(x)\|_A\le\frac{1}{2^2.7}\sum_{i=\frac{1-j}{2}}^{\infty}\frac{\alpha\left(\left(\sqrt{2}\right)^{ij}\sqrt{x},\dots,\left(\sqrt{2}\right)^{ij}\sqrt{x}\right)}{2^{2ij}} \tag{4.21}$$

for all $x\in\mathrm{A}$. The mapping $\mathrm{D}(x)$ is defined by

$$\mathrm{D}(x)=\lim_{k\to\infty}\frac{f(2^{kj}x)}{2^{2kj}} \tag{4.22}$$

for all $x\in\mathrm{A}$.

Proof. Assume $j=1$. By the same reasoning as that in the proof of the Theorem 4.2, there exists a unique quadratic mapping $\mathrm{D}:\mathrm{A}\to\mathrm{A}$ satisfying (4.21). The mapping $\mathrm{D}:\mathrm{A}\to\mathrm{A}$ given by $\mathrm{D}(x)=\lim_{k\to\infty}\frac{f(2^{kj}x)}{2^{2kj}}$. It follows from (4.19) that

$$\left\|\mathrm{D}(x_1y_1)-x_1^2\,\mathrm{D}(y_1)-\mathrm{D}(x_1)y_1^2\right\|_A=\lim_{k\to\infty}\frac{1}{2^{4kj}}\left\|f\left(2^{2k}x_1y_1\right)-\left(2^kx_1\right)^2f\left(2^ky_1\right)-f\left(2^kx_1\right)\left(2^ky_1\right)^2\right\|_A$$
$$\le\lim_{k\to\infty}\frac{1}{2^{4kj}}\alpha\left(2^kx_1,2^ky_1,0,0,\dots,0,0\right)=0$$

for all $x_1,y_1\in\mathrm{A}$. Therefore $\mathrm{D}:\mathrm{A}\to\mathrm{A}$ is a quadratic derivation satisfying (4.21).

The following corollary is an immediate consequence of Theorem 4.5 concerning the stability of (1.5).

Corollary 4.6 *Let* λ *and* s *be non-negative real numbers. If a function* $f : \mathrm{A} \to \mathrm{A}$ *satisfies the inequality*

$$\left\| \mathrm{F}(x_1, y_1, \ldots, x_8, y_8) \right\|_A \leq \begin{cases} \lambda, \\ \lambda \sum_{i=1}^{8} \left\{ \| x_i \|_A^s + \| y_i \|_A^s \right\}, \\ \lambda \left\{ \prod_{i=1}^{8} \| x_i \|_A^s \| y_i \|_A^s \right\}, \\ \lambda \left\{ \prod_{i=1}^{8} \| x_i \|_A^s \| y_i \|_A^s + \sum_{i=1}^{8} \left(\| x_i \|_A^{16s} + \| y_i \|_A^{16s} \right) \right\}, \end{cases} \tag{4.23}$$

for all $x_1, y_1, \ldots, x_8, y_8 \in \mathrm{A}$ and

$$\left\| f(xy) - x^2 f(y) - f(x) y^2 \right\|_A \leq \begin{cases} \lambda, \\ \lambda \left\{ \| x \|_A^s + \| y \|_A^s \right\}, \\ \lambda \left\{ \| x \|_A^s \| y \|_A^s \right\}, \\ \lambda \left\{ \| x \|_A^s \| y \|_A^s + \left(\| x \|_A^{2s} + \| y \|_A^{2s} \right) \right\} \end{cases} \tag{4.24}$$

for all $x, y \in \mathrm{A}$. Then there exists a unique quadratic derivation $\mathrm{D} : \mathrm{A} \to \mathrm{A}$ such that

$$\left\| f(x) - \mathrm{D}(x) \right\|_A \leq \begin{cases} \dfrac{\lambda}{21}; \\ \dfrac{16\lambda \| x \|_A^{s/2}}{7 \left| 2^2 - 2^{\frac{s}{2}} \right|}, & s \neq 4; \\ \dfrac{\lambda \| x \|_A^{8s}}{7 \left| 2^2 - 2^{8s} \right|}, & s \neq \dfrac{1}{4}; \\ \dfrac{17\lambda \| x \|_A^{8s}}{7 \left| 2^2 - 2^{8s} \right|}, & s \neq \dfrac{1}{4} \end{cases} \tag{4.25}$$

for all $x \in \mathrm{A}$.

5. Stability of Quadratic Homomorphisms and Quadratic Derivations of (1.5): Fixed Point Method

In this section, the authors presented the generalized Ulam-Hyers stability of the functional equation (1.5) in quasi-Banach algebra using fixed point method.

Now we will recall the fundamental results in fixed point theory.

Theorem 5.1 *(Banach Contraction Principle)*

Let (Ω, d) *is a non-Archimedean generalized complete metric space and consider a mapping* $T:\Omega \to \Omega$ *which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$, for all $x, y \in T$ for some (Lipschitz constant) $L < 1$. Then,

(i) The mapping T has one and only fixed point $x^* = T(x^*)$;

(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty} T^n x = x^*$, for any starting point $x \in \Omega$;

(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \dfrac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in \Omega;$

(A4) $d(x, x^*) \leq \dfrac{1}{1-L}\ d(x, x^*), \forall\ x \in \Omega..$

Theorem 5.2 *[13](The alternative of fixed point)*

Suppose that for a complete generalized metric space (A, d) *and a strictly contractive mapping* $T:\mathrm{A} \to \mathrm{A}$ *with Lipschitz constant* L. *Then, for each given element* $x \in \mathrm{A}$, *either*

$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \geq 0,$$

or

$(B2)$ there exists a natural number n_0 such that:

(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in \mathrm{A} : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \dfrac{1}{1-L}\ d(y, Ty)$ for all $y \in Y$.

Theorem 5.3

Let $f:\mathrm{A} \to \mathrm{B}$ *be a mapping for which there exists a function* $\varphi:\mathrm{A}^{16} \to [0,\infty)$ *with the condition*

$$\lim_{k\to\infty}\frac{1}{\mu_i^{2k}}\alpha(\mu_i^k x_1,\mu_i^k y_1,\ldots,\mu_i^k x_8,\mu_i^k y_8)=0 \tag{5.1}$$

where $\mu_i=2$ if $i=0$ and $\mu_i=\frac{1}{2}$ if $i=1$ such that the functional inequality with

$$\|\mathrm{F}(x_1,y_1,\ldots,x_8,y_8)\|_B\le\alpha(x_1,y_1,\ldots,x_8,y_8) \tag{5.2}$$

for all $x_1,y_1,\ldots,x_8,y_8\in\mathrm{A}$ and

$$\|f(xy)-f(x)f(y)\|_B\le\alpha\left(x,y,0,0,\ldots,0,0\right) \tag{5.3}$$

for all $x,y\in\mathrm{A}$. If there exists $L=L(i)<1$ such that the function

$$x\to\gamma(x)=\frac{1}{7}\alpha\left(\sqrt{\frac{x}{2}},\sqrt{\frac{x}{2}},\ldots,\sqrt{\frac{x}{2}},\sqrt{\frac{x}{2}}\right) \tag{5.4}$$

has the property

$$\gamma(x)=L\frac{1}{\mu_i^2}\gamma\left(\mu_i x\right), \tag{5.5}$$

then there exists a unique quadratic homomorphism $\mathrm{H}:\mathrm{A}\to\mathrm{B}$ satisfying the functional equation (1.5) and

$$\|f(x)-\mathrm{H}(x)\|_B\le\frac{L^{1-i}}{1-L}\gamma(x) \tag{5.6}$$

for all $x\in\mathrm{A}$.

Proof. Consider the set

$$\Omega=\{p/p:\mathrm{A}\to\mathrm{B},\ p(0)=0\}$$

and introduce the generalized metric on Ω,

$$d(p,q)=\inf\{K\in(0,\infty):\|p(x)-q(x)\|_B\le K\gamma(x),x\in\mathrm{A}\}.$$

It is easy to see that (Ω,d) is complete.

Define $T:\Omega\to\Omega$ by $\qquad Tp(x)=\frac{1}{\mu_i^2}p(\mu_i x),$

for all $x\in\mathrm{A}$. Now $p,q\in\Omega$, we have

$$d(p,q)\le K\Rightarrow\|p(x)-q(x)\|_B\le K\gamma(x),x\in\mathrm{A}.$$

$$\Rightarrow \left\| \frac{1}{\mu_i^2} p(\mu_i x) - \frac{1}{\mu_i^2} q(\mu_i x) \right\|_B \leq \frac{1}{\mu_i^2} K\gamma(\mu_i x), x \in \mathrm{A},$$

$$\Rightarrow \left\| \frac{1}{\mu_i^2} p(\mu_i x) - \frac{1}{\mu_i^2} q(\mu_i x) \right\| \leq LK\gamma(x), x \in \mathrm{A},$$

$$\Rightarrow \| Tp(x) - Tq(x) \|_B \leq LK\gamma(x), x \in \mathrm{A},$$

$$\Rightarrow d(p,q) \leq LK.$$

This implies $d(Tp,Tq) \leq Ld(p,q)$, for all $p,q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L .

Replacing $(x_1, y_1, \ldots, x_8, y_8)$ by $\left(\sqrt{x}, \sqrt{x}, \ldots, \sqrt{x}, \sqrt{x}\right)$ and dividing by 28 in (5.2), we get

$$\| f(2x) - 2^2 f(x) \|_B \leq \frac{1}{7} \alpha\left(\sqrt{x}, \sqrt{x}, \ldots, \sqrt{x}, \sqrt{x}\right) \tag{5.7}$$

for all $x \in \mathrm{A}$. Hence from the above inequality, we have

$$\left\| \frac{f(2x)}{2^2} - f(x) \right\|_B \leq \frac{1}{2^2.7} \alpha\left(\sqrt{x}, \sqrt{x}, \ldots, \sqrt{x}, \sqrt{x}\right) \tag{5.8}$$

for all $x \in \mathrm{A}$. Using (5.4) and (5.5) for the case $i = 0$, it reduces to

$$\left\| \frac{f(2x)}{2^2} - f(x) \right\|_B \leq \frac{1}{2^2} \gamma(x)$$

for all $x \in \mathrm{A}$.

$$\text{i.e.,} \quad d(f, Tf) \leq L \leq L^1 < \infty.$$

Again replacing x by $\frac{x}{2}$ in (5.7), we get

$$\left\| f(x) - 2^2 f\left(\frac{x}{2}\right) \right\|_B \leq \frac{1}{7} \alpha\left(\sqrt{\frac{x}{2}}, \sqrt{\frac{x}{2}}, \ldots, \sqrt{\frac{x}{2}}, \sqrt{\frac{x}{2}}\right) \tag{5.9}$$

for all $x \in \mathrm{A}$. Using (5.4) and (5.5) for the case $i = 1$ it reduces to

$$\left\| f(x) - 2^2 f\left(\frac{x}{2}\right) \right\|_B \leq \gamma(x)$$

for all $x \in \mathrm{A}$,

$$i.e., \quad d(f,Tf) \le 1 \le L^0 < \infty.$$

In both cases, we arrive

$$d(f,Tf) \le L^{1-i}.$$

Therefore $(A1)$ holds.

By $(A2)$, it follows that there exists a fixed point H of T in Ω such that

$$\mathrm{H}(x) = \lim_{k\to\infty} \frac{1}{\mu_i^{2k}} f(\mu_i^k x) \tag{5.10}$$

for all $x \in \mathrm{A}$.

To prove $\mathrm{H}:\mathrm{A} \to \mathrm{B}$ is quadratic. Replacing $(x_1, y_1, \dots, x_8, y_8)$ by $\left(\mu_i^k x_1, \mu_i^k y_1, \dots, \mu_i^k x_8, \mu_i^k y_8\right)$ in (5.2) and dividing by μ_i^k, it follows from (5.1) that

$$\|\mathrm{H}(x_1, y_1, \dots, x_8, y_8)\|_B = \lim_{k\to\infty} \frac{\|\mathrm{F}(\mu_i^k x_1, \mu_i^k y_1, \dots, \mu_i^k x_8, \mu_i^k y_8)\|_B}{\mu_i^{2k}}$$

$$\le \lim_{k\to\infty} \frac{\alpha\left(\mu_i^k x_1, \mu_i^k y_1, \dots, \mu_i^k x_8, \mu_i^k y_8\right)}{\mu_i^{2k}} = 0$$

for all $x_1, y_1, \dots, x_8, y_8 \in \mathrm{A}$. i.e., H satisfies the functional equation (1.5). Also,

$$\|\mathrm{H}(x_1 y_1) - \mathrm{H}(x_1)\,\mathrm{H}(y_1)\|_B \le \lim_{k\to\infty} \frac{1}{\mu_i^{4k}} \left\| f\left(\mu_i^{2k} x_1 y_1\right) - f\left(\mu_i^k x_1\right) f\left(\mu_i^k y_1\right) \right\|_B$$

$$\le \lim_{k\to\infty} \frac{1}{\mu_i^{4k}} \alpha\left(\mu_i^k x_1, 0, \mu_i^k y_1, 0\right) = 0$$

for all $x_1, y_1 \in \mathrm{A}$. Therefore, H is a quadratic homomorphism.

By $(A3)$, H is the unique fixed point of T in the set $\Delta = \{\mathrm{H} \in \Omega : d(f,\mathrm{H}) < \infty\}, \mathrm{H}$ is the unique function such that

$$\| f(x) - \mathrm{H}(x)\|_B \le K\gamma(x)$$

for all $x \in \mathrm{A}$ and $K > 0$. Finally by $(A4)$, we obtain

$$d(f,\mathrm{H}) \le \frac{1}{1-L} d(f,Tf)$$

this implies

$$d(f,\mathrm{H}) \le \frac{L^{1-i}}{1-L}$$

which yields

$$\| f(x) - \mathrm{H}(x) \|_B \le \frac{L^{1-i}}{1-L}\gamma(x)$$

this completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 5.3 concerning the stability of (1.5).

Corollary 5.4 *Let* $f:\mathrm{A}\to\mathrm{B}$ *be a mapping and there exists real numbers* λ *and* s *such that*

$$\|\mathrm{F}(x_1,y_1,\ldots,x_8,y_8)\|_B \le \begin{cases} \lambda, \\ \lambda\sum_{i=1}^{8}\left\{\|x_i\|_A^s + \|y_i\|_A^s\right\}, \\ \lambda\left\{\prod_{i=1}^{8}\|x_i\|_A^s\|y_i\|_A^s\right\}, \\ \lambda\left\{\prod_{i=1}^{8}\|x_i\|_A^s\|y_i\|_A^s + \sum_{i=1}^{8}\left(\|x_i\|_A^{16s} + \|y_i\|_A^{16s}\right)\right\}, \end{cases} \tag{5.11}$$

for all $x_1,y_1,\ldots,x_8,y_8 \in \mathrm{A}$ and

$$\|f(x,y) - f(x)f(y)\|_B \le \begin{cases} \lambda, \\ \lambda\left\{\|x\|_A^s + \|y\|_A^s\right\}, \\ \lambda\left\{\|x\|_A^s\|y\|_A^s\right\}, \\ \lambda\left\{\|x\|_A^s\|y\|_A^s + \left(\|x\|_A^{2s} + \|y\|_A^{2s}\right)\right\} \end{cases} \tag{5.12}$$

for all $x,y\in\mathrm{A}$. Then there exists a unique quadratic homomorphism $\mathrm{H}:\mathrm{A}\to\mathrm{B}$ such that

$$\| f(x) - \mathrm{H}(x) \|_B \leq \begin{cases} \dfrac{\lambda}{21}, & \\ \dfrac{16\lambda \| x \|_A^{s/2}}{7 \, | 2^2 - 2^{\frac{s}{2}} |}, & s \neq 4; \\ \dfrac{\lambda \| x \|_A^{8s}}{7 \, | 2^2 - 2^{8s} |}, & s \neq \dfrac{1}{4}; \\ \dfrac{17\lambda \| x \|_A^{8s}}{7 \, | 2^2 - 2^{8s} |}, & s \neq \dfrac{1}{4} \end{cases} \tag{5.13}$$

for all $x \in \mathrm{A}$.

Proof. Setting

$$\alpha(x_1, y_1, \ldots, x_8, y_8) = \begin{cases} \lambda, \\ \lambda \displaystyle\sum_{i=1}^{8} \left\{ \| x_i \|_A^s + \| y_i \|_A^s \right\}, \\ \lambda \left\{ \displaystyle\prod_{i=1}^{8} \| x_i \|_A^s \| y_i \|_A^s \right\}, \\ \lambda \left\{ \displaystyle\prod_{i=1}^{8} \| x_i \|_A^s \| y_i \|_A^s + \sum_{i=1}^{8} \left(\| x_i \|_A^{16s} + \| y_i \|_A^{16s} \right) \right\}, \end{cases}$$

for all $x_1, y_1, \ldots, x_8, y_8 \in \mathrm{A}$. Now,

$$\frac{\alpha(\mu_i^k x_1, \mu_i^k y_1, \ldots, \mu_i^k x_8, \mu_i^k y_8)}{\mu_i^{2k}} = \begin{cases} \dfrac{\lambda}{\mu_i^{2k}}, \\ \dfrac{\lambda}{\mu_i^{2k}} \displaystyle\sum_{i=1}^{8} \left\{ \| \mu_i^k x_i \|_A^s + \| \mu_i^k y_i \|_A^s \right\}, \\ \dfrac{\lambda}{\mu_i^{2k}} \left\{ \displaystyle\prod_{i=1}^{8} \| \mu_i^k x_i \|_A^s \| \mu_i^k y_i \|_A^s \right\}, \\ \dfrac{\lambda}{\mu_i^{2k}} \left\{ \displaystyle\prod_{i=1}^{8} \| \mu_i^k x_i \|_A^s \| \mu_i^k y_i \|_A^s + \sum_{i=1}^{8} \left(\| \mu_i^k x_i \|_A^{16s} + \| \mu_i^k y_i \|_A^{16s} \right) \right\}, \end{cases}$$

$$= \begin{cases} \lambda \mu_i^{-2k}, \\ \lambda \mu_i^{(s-2)k} \sum_{i=1}^{8} \left\{ \| x_i \|_A^s + \| y_i \|_A^s \right\}, \\ \lambda \mu_i^{(16s-2)k} \left\{ \prod_{i=1}^{8} \| x_i \|_A^s \| y_i \|_A^s \right\}, \\ \lambda \mu_i^{(16s-2)k} \left\{ \prod_{i=1}^{8} \| x_i \|_A^s \| y_i \|_A^s + \sum_{i=1}^{8} \left(\| x_i \|_A^{16s} + \| y_i \|_A^{16s} \right) \right\}, \end{cases}$$

$$= \begin{cases} \to 0 \; as \;\; k \to \infty, \\ \to 0 \; as \;\; k \to \infty, \\ \to 0 \; as \;\; k \to \infty, \\ \to 0 \; as \;\; k \to \infty. \end{cases}$$

Thus, (5.1) is holds.

But we have $\gamma(x) = \frac{1}{7} \alpha \left(\sqrt{\frac{x}{2}}, \sqrt{\frac{x}{2}}, \ldots, \sqrt{\frac{x}{2}}, \sqrt{\frac{x}{2}} \right)$ has the property $\gamma(x) = L \cdot \frac{1}{\mu_i^2} \gamma(\mu_i x)$ for all $x \in \mathrm{A}$. Hence

$$\gamma(x) = \frac{1}{7} \alpha \left(\sqrt{\frac{x}{2}}, \sqrt{\frac{x}{2}}, \ldots, \sqrt{\frac{x}{2}}, \sqrt{\frac{x}{2}} \right) = \begin{cases} \frac{\lambda}{7}, \\ \frac{16\lambda}{7\left(2^{\frac{s}{2}} \right)} \| x \|_A^{\frac{s}{2}}, \\ \frac{\lambda}{7\left(2^{8s} \right)} \| x \|_A^{8s}, \\ \frac{\lambda}{7} \left(\frac{1}{2^{8s}} + \frac{16}{2^{8s}} \right) \| x \|_A^{8s} . \end{cases}$$

Now,

$$\frac{1}{\mu_i^2}\gamma(\mu_i x) = \begin{cases} \dfrac{\lambda}{7\mu_i^2}, \\ \dfrac{16\lambda}{7\left(2^{\frac{s}{2}}\right)\mu_i^2}\|\mu_i x\|_A^{\frac{s}{2}}, \\ \dfrac{\lambda}{7\left(2^{8s}\right)\mu_i^2}\|\mu_i x\|_A^{8s}, \\ \dfrac{\lambda}{7\mu_i^2}\left(\dfrac{1}{2^{8s}}+\dfrac{16}{2^{8s}}\right)\|\mu_i x\|_A^{8s} \end{cases} = \begin{cases} \mu_i^{-2}\gamma(x), \\ \mu_i^{\frac{s-4}{2}}\gamma(x), \\ \mu_i^{8s-2}\gamma(x), \\ \mu_i^{8s-2}\gamma(x). \end{cases}$$

From (5.6), we prove the following six cases:

Case 1: If $i = 0$ then $L = 2^{-2}$

$$\left\| f(x) - \mathrm{H}(x) \right\|_B \le \frac{\left(2^{-2}\right)^{1-0}}{1-2^{(-2)}}\gamma(x) = \frac{\lambda}{7}\left(\frac{2^{-2}}{1-2^{-2}}\right) = \frac{\lambda}{21}.$$

Case 2: If $i = 1$ then $L = 2^2$

$$\left\| f(x) - \mathrm{H}(x) \right\|_B \le \frac{\left(2^{2}\right)^{1-1}}{1-2^{2}}\gamma(x) = \frac{\lambda}{7}\left(\frac{1}{1-2^{2}}\right) = \frac{\lambda}{-21}.$$

Case 3: $L = 2^{(s-4)/2}$ for $s < 4$ if $i = 0$

$$\left\| f(x) - \mathrm{H}(x) \right\|_B \le \frac{\left(2^{(s-4)/2}\right)^{1-0}}{1-2^{(s-4)/2}}\gamma(x) = \frac{16\lambda}{7\left(2^2 - 2^{\frac{s}{2}}\right)}\|x\|_A^{\frac{s}{2}}.$$

Case 4: $L = 2^{(4-s)/2}$ for $s > 4$ if $i = 1$

$$\left\| f(x) - \mathrm{H}(x) \right\|_B \le \frac{\left(2^{(4-s)/2}\right)^{1-1}}{1-2^{(4-s)/2}}\gamma(x) = \frac{16\lambda}{7\left(2^{\frac{s}{2}} - 2^2\right)}\|x\|_A^{\frac{s}{2}}.$$

Case 5: $L = 2^{8s-2}$ for $s < \frac{1}{4}$ if $i = 0$

$$\left\| f(x) - \mathrm{H}(x) \right\|_B \le \frac{\left(2^{(8s-2)}\right)^{1-0}}{1-2^{(8s-2)}}\gamma(x) = \frac{\lambda}{7\left(2^2 - 2^{8s}\right)}\|x\|_A^{8s}.$$

Case 6: $L = 2^{2-8s}$ for $s > \frac{1}{4}$ if $i = 1$

$$\| f(x) - \mathrm{H}(x) \|_B \le \frac{\left(2^{(2-8s)}\right)^{1-1}}{1 - 2^{(2-8s)}} \gamma(x) = \frac{\lambda}{7\left(2^{8s} - 2^2\right)} \| x \|_A^{8s} \, .$$

Hence the proof is complete

Theorem 5.5

Let $f : \mathrm{A} \to \mathrm{A}$ *be a mapping for which there exists a function* $\alpha : \mathrm{A}^{16} \to [0, \infty)$ *with the condition*

$$\lim_{k\to\infty} \frac{1}{\mu_i^{2k}} \alpha(\mu_i^k x_1, \mu_i^k y_1, \ldots, \mu_i^k x_8, \mu_i^k y_8) = 0 \tag{5.14}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality with

$$\left\| \mathrm{F}(x_1, y_1, \ldots, x_8, y_8) \right\|_A \le \alpha(x_1, y_1, \ldots, x_8, y_8) \tag{5.15}$$

for all $x_1, y_1, \ldots, x_8, y_8 \in \mathrm{A}$ and

$$\left\| f(xy) - x^2 f(y) - f(x) y^2 \right\|_A \le \alpha\left(x, y, 0, 0, \ldots, 0, 0\right) \tag{5.16}$$

for all $x, y \in \mathrm{A}$. If there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \alpha\left(\sqrt{\frac{x}{2}}, \sqrt{\frac{x}{2}}, \ldots, \sqrt{\frac{x}{2}}, \sqrt{\frac{x}{2}} \right), \tag{5.17}$$

has the property

$$\gamma(x) = L \frac{1}{\mu_i^2} \gamma\left(\mu_i x\right) \tag{5.18}$$

for all $x \in \mathrm{A}$, then there exists a unique quadratic derivation $\mathrm{D} : \mathrm{A} \to \mathrm{A}$ satisfying the functional equation (1.5) and

$$\| f(x) - \mathrm{D}(x) \|_A \le \frac{L^{1-i}}{1 - L} \gamma(x) \tag{5.19}$$

for all $x \in \mathrm{A}$.

Proof. By the same reasoning as that in the proof of Theorem 5.3, there exists a unique quadratic mapping $\mathrm{D}:\mathrm{A}\to\mathrm{A}$ satisfying (5.19). The mapping $\mathrm{D}:\mathrm{A}\to\mathrm{A}$ is given by $\mathrm{D}(x) = \lim\limits_{k\to\infty}\dfrac{f\left(\mu_i^k x\right)}{\mu_i^{2k}}$ for all $x\in\mathrm{A}$. It follows from (5.15) that

$$\left\|\mathrm{D}(x_1y_1)-x_1^2\,\mathrm{D}(y_1)-\mathrm{D}(x_1)y_1^2\right\|_A \le \lim_{k\to\infty}\frac{1}{\mu_i^{4k}}\left\| f\left(\mu_i^{2k}x_1y_1\right)-\left(\mu_i x_1\right)^2 f\left(\mu_i^k y_1\right)-f\left(\mu_i^k x_1\right)\left(\mu_i^k y_1\right)^2\right\|_A$$

$$\le \lim_{k\to\infty}\frac{1}{\mu_i^{4k}}\,\alpha\left(\mu_i^k x_1,\mu_i^k y_1,0,0,\ldots,0,0\right)=0$$

for all $x_1, y_1\in\mathrm{A}$. Therefore, $\mathrm{D}:\mathrm{A}\to\mathrm{A}$ is a quadratic derivation satisfying. The rest of the proof is similar to that of Theorem 5.3

The following corollary is an immediate consequence of Theorem 5.5 concerning the stability of (1.5).

Corollary 5.6 *Let* λ *and* s *be non-negative real numbers. If a function* $f:\mathrm{A}\to\mathrm{A}$ *satisfies the inequality*

$$\left\|\mathrm{F}(x_1,y_1,\ldots,x_8,y_8)\right\|_A \le \begin{cases} \lambda, \\ \lambda\sum\limits_{i=1}^{8}\left\{\|x_i\|_A^s+\|y_i\|_A^s\right\}, \\ \lambda\left\{\prod\limits_{i=1}^{8}\|x_i\|_A^s\|y_i\|_A^s\right\}, \\ \lambda\left\{\prod\limits_{i=1}^{8}\|x_i\|_A^s\|y_i\|_A^s+\sum\limits_{i=1}^{8}\left(\|x_i\|_A^{16s}+\|y_i\|_A^{16s}\right)\right\}, \end{cases} \tag{5.20}$$

for all $x_1,y_1,\ldots,x_8,y_8\in\mathrm{A}$ and

$$\left\|f(xy)-x^2f(y)-f(x)y^2\right\|_A \le \begin{cases} \lambda, \\ \lambda\left\{\|x\|_A^s+\|y\|_A^s\right\}, \\ \lambda\left\{\|x\|_A^s\|y\|_A^s\right\}, \\ \lambda\left\{\|x\|_A^s\|y\|_A^s+\left(\|x\|_A^{2s}+\|y\|_A^{2s}\right)\right\} \end{cases} \tag{5.21}$$

for all $x, y\in\mathrm{A}$. Then there exists a unique quadratic derivation $\mathrm{D}:\mathrm{A}\to\mathrm{A}$ such that

$$\left\| f(x) - \mathrm{D}(x) \right\|_A \le \begin{cases} \dfrac{\lambda}{21}; \\ \dfrac{16\lambda \, \| x \|_A^{s/2}}{7 \, | 2^2 - 2^{\frac{s}{2}} |}, & s \ne 4; \\ \dfrac{\lambda \, \| x \|_A^{8s}}{7 \, | 2^2 - 2^{8s} |}, & s \ne \dfrac{1}{4}; \\ \dfrac{17\lambda \, \| x \|_A^{8s}}{7 \, | 2^2 - 2^{8s} |}, & s \ne \dfrac{1}{4} \end{cases} \tag{5.22}$$

for all $x \in \mathrm{A}$.

6. Applications of The Functional Equation (1.5)

Consider the functional equation (1.5), that is

$$\begin{aligned} \left(\sum_{i=1}^{8} f(x_i) \right) \left(\sum_{i=1}^{8} f(y_i) \right) = {} & f\left(x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8 \right) \\ & + f\left(x_2y_1 + x_1y_2 + x_4y_3 - x_3y_4 + x_6y_5 - x_5y_6 + x_8y_7 - x_7y_8 \right) \\ & + f\left(x_3y_1 - x_4y_2 + x_1y_3 + x_2y_4 + x_7y_5 + x_8y_6 - x_5y_7 - x_6y_8 \right) \\ & + f\left(x_4y_1 + x_3y_2 - x_2y_3 + x_1y_4 + x_8y_5 - x_7y_6 + x_6y_7 - x_5y_8 \right) \\ & + f\left(x_5y_1 - x_6y_2 - x_7y_3 - x_8y_4 + x_1y_5 + x_2y_6 + x_3y_7 + x_4y_8 \right) \\ & + f\left(x_6y_1 + x_5y_2 - x_8y_3 + x_7y_4 - x_2y_5 + x_1y_6 - x_4y_7 + x_3y_8 \right) \\ & + f\left(x_7y_1 + x_8y_2 + x_5y_3 - x_6y_4 - x_3y_5 + x_4y_6 + x_1y_7 - x_2y_8 \right) \\ & + f\left(x_8y_1 - x_7y_2 + x_6y_3 + x_5y_4 - x_4y_5 - x_3y_6 + x_2y_7 + x_1y_8 \right) \end{aligned}$$

Since $f(x) = x^2$ is the solution of the above functional equation, then the above equation can be written as follows

$$\begin{aligned}\left(\sum_{i=1}^{8}(x_i)^2\right)\left(\sum_{i=1}^{8}(y_i)^2\right) &= \left(x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8\right)^2 \\ &+ \left(x_2y_1 + x_1y_2 + x_4y_3 - x_3y_4 + x_6y_5 - x_5y_6 + x_8y_7 - x_7y_8\right)^2 \\ &+ \left(x_3y_1 - x_4y_2 + x_1y_3 + x_2y_4 + x_7y_5 + x_8y_6 - x_5y_7 - x_6y_8\right)^2 \\ &+ \left(x_4y_1 + x_3y_2 - x_2y_3 + x_1y_4 + x_8y_5 - x_7y_6 + x_6y_7 - x_5y_8\right)^2 \\ &+ \left(x_5y_1 - x_6y_2 - x_7y_3 - x_8y_4 + x_1y_5 + x_2y_6 + x_3y_7 + x_4y_8\right)^2 \\ &+ \left(x_6y_1 + x_5y_2 - x_8y_3 + x_7y_4 - x_2y_5 + x_1y_6 - x_4y_7 + x_3y_8\right)^2 \\ &+ \left(x_7y_1 + x_8y_2 + x_5y_3 - x_6y_4 - x_3y_5 + x_4y_6 + x_1y_7 - x_2y_8\right)^2 \\ &+ \left(x_8y_1 - x_7y_2 + x_6y_3 + x_5y_4 - x_4y_5 - x_3y_6 + x_2y_7 + x_1y_8\right)^2\end{aligned}$$

The above identity is called Degen-Graves-Cayley-Eight Squares identity and shows that “the product of two sums of eight squares is again a sum of eight squares".

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