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A NON MARKOVIAN RETRIAL QUEUEINGSYSTEM WITH MODIFIED EXTENDED VACATIONS UNDER BERNOULLI SCHEDULE AND SERVER BREAKDOWNS

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ABSTRACT:

This paper deals with the steady state behaviour of a batch arrival retrial queueing system with two phases of services, modified extended vacations under Bernoulli schedule, where the server is subject to breakdowns and repairs. Any arriving batch finding the server busy, breakdown or on vacation enters an orbit. Otherwise one customer from the arriving batch enters the service immediately while the rest join the orbit. All customers demand the first 'essential' service, whereas only some of them demand the second multiple 'optional' service. After every service completion the server may go on an optional vacation with probability p or may wait for serving the next customer with probability (1-p). When the original vacation is completed the server has the option to go on an extended vacation. The server is working with any phase of service, it may breakdown at any instant and the service channel will fail for a short interval of time. We construct the mathematical model and derive the probability generating functions of number of customers in the systemby using the supplementary variable method. Some system performances and special cases are obtained.

Keywords: Retrial queue, two phase service, modified extended vacations,breakdowns and repairs.

1. Introduction

Recently there have been significant contributions to retrial queueing system in which have wide applications in telephone switching systems, telecommunication networks and computers communications (Falin and Templeton, 1997 and Artalejo, 1999). Queueing systems with optional serviceare characterized by some of the authors like Artalejo and Choudhury (2004), Wang and Li (2009), Ke and Chang (2009). Krishnakumar and Arivudainambi (2002) have investigated a single server retrial queue with Bernoulli schedule and general retrial times.Comprehensive surveys on vacations can be found in Doshi (1986). Ke (2007) studied the variant vacation policy in M/G/1 queue.Choudhury et al. (2010) discussed about the batch arrival retrial queueing system with two phases of service under the concept of breakdown.Choudhury and Deka (2008) considered a single server queue with two phases of service and the server is subject to breakdown while providing service to the customers.In this paper, we discussed a model $M^{[X]}/G/1$ retrial queue with second optional service, modifiedextended vacations under Bernoulli schedule and repair. The suggested model has potential application in the transfer model of an email system. Simple mail transfer protocol (SMTP) is used to deliver the messages between the mail servers.

2. Description of the Model

In this section, we develop a model for batch arrival retrial queue with second optional service, modified extended vacations under Bernoulli schedule, where the server is subject to breakdowns and repair. The detailed description of the model is given as follows:

***Arrival process*:** Customers arrive in batches according to a compound Poisson process with rateλ. Let X_u denote the number of customers belonging to the u^{th} arrival batch, where X_u , u =1,2 ,3,…are with a common distribution $\Pr[X_u = n] = \chi_n$, n = 1,2,3… and $X(z)$ denotes the probability generating function of X.

***Retrial process*:** We assume that there is no waiting space and therefore if an arriving batch of customers finds the server free, the arrival beings his service one from the batch and rest of them join into pool of blocked customers called an orbit. If an arriving batch finds the server being busy, vacation or breakdown, the arrivals leave the service area and join into an orbit. Inter-retrial times have an arbitrary distribution $R(t)$ with corresponding Laplace-Stieltijes transform (LST) $R^*(\vartheta)$.

***Service process*:** The First Essential Service (FES) is needed to all arriving customers and the service time has a general distribution. The service time of the first essential service is denoted by the random variable S_0 with distribution function $S_0(t)$ and LST $S_0^*(t)$. As soon as first essential service completed, with probability r_k ($1 \leq k \leq m$), the customer may chose for a certain Second Optional Service (SOS) from m ($m \geq 1$) kinds of different services, or else with probability $r_0 = 1 - \sum_{k=1}^{m} r_k$ he leaves the system. The service times follows a general random variable S_k with d.f $S_k(t)$ and LST $S_k^*(\vartheta)$.

***Vacation process*:**After completion of service to each customer, the server may take a vacationwith probability p and with probability $q = 1-p$ it waits for the next customer to serve. If orbit is empty, the server always takes a vacation. At the end of a vacation, the server remains idle for the customer from the orbit or new arrival customers. The vacation time of the server is of random length V with distribution function $V(t)$ and LST$V^*(\vartheta)$.

Breakdownprocess:While the server is working with any phase of service, it may breakdown at any time and the service channel will fail for a short interval of time i.e. server is down for a short interval of time. The breakdowns i.e. server's life times are generated by exogenous Poisson processes with rates α_0 for FES and α_k for SOS, which we may call some sort of disaster during FES and SOS periods respectively.

Repairprocess: As soon as breakdown occurs the server is sent for repair, during that time it stops providing service to the arriving batch of customers. The customer who was just being served before server breakdown waits for the remaining service to complete. The repair time (denoted by G_0 for FES and G_k for SOS) distributions of the server for both the phases of service are assumed to be arbitrarily distributed with d.f.$G_0(y)$ for FES and $G_k(y)$ for SOS. It is LST$G_0^*(\vartheta)$ and $G_k^*(\vartheta)$.

Various stochastic processes involved in the system are assumed to be independent of each other.

In the steady state, we assume that $R(0)=0$, $R(\infty)=1$, $S_0(0)=0$, $S_0(\infty)=1$ $S_k(0)=0$, $S_k(\infty)=1$, $V(0)=0$, $V(\infty)=1$are continuous at $x = 0$ and $G_0(0)=0$, $G_k(\infty)=1$ are continuous at $y = 0$ ($1 \leq k \leq m$).The state of the system at time $t\, R^0(t), S_0^0(t), S_k^0(t), V^0(t), G_0^0(t)$ and $G_k^0(t)$ be the elapsed times respectively, the elapsed service time of the FES, service time of the SOS, vacation time, repair time on FES and repair time on SOS.

Further, introduce the random variables

$$C(t)=\begin{cases}0,\ \text{if the server is idle time } t,\\ 1,\ \text{if the server is busy on FES time } t,\\ 2,\ \text{if the server is busy on SOS time } t,\\ 3,\ \text{if the server is on vacation time } t,\\ 4,\ \text{if the server is repair on FES time } t,\\ 5,\ \text{if the server is repair on SOS time } t.\end{cases}$$

The state of system at time t can be described by the bivariate Markov process $\{C(t), N(t); t\geq 0\}$ where $C(t)$ denotes the server state (0,1,2,3,4,5) depending if the server is idle, busy on FES or SOS, vacation and repair on FES or SOS respectively. $N(t)$ corresponding to the number of customers in orbit at time t.

So that the functions $a(x),\ \mu_0(x),\ \mu_k(x),\ \gamma(x),\ \xi_0(y)$ and $\xi_k(y)$ are the conditional completion rates for repeated attempts, service, vacation and repair time respectively ($1\leq k\leq m$). Conditional completion rates for repeated attempts, service on phases, vacation, repair on server and repair on phases respectively ($1\leq k\leq m$),

$$a(x)dx=\frac{dR(x)}{1-R(x)};\ \mu_0(x)dx=\frac{dS_0(x)}{1-S_0(x)};\ \mu_k(x)dx=\frac{dS_k(x)}{1-S_k(x)};$$
$$\gamma(x)dx=\frac{dV(x)}{1-V(x)};\ \xi_0(y)dy=\frac{dG_0(y)}{1-G_0(y)};\ \xi_k(y)dy=\frac{dG_k(y)}{1-G_k(y)};\ ,$$

Let $\{t_n;\ n = 1,2,\ldots\}$ be the sequence of epochs at which either a service period completion occurs or a vacation time ends. The sequence of random vectors $Z_n = \{C(t_n+),\ N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system.

Theorem 2.1: The embedded Markov chain $\{Z_n;\ n\in N\}$ is ergodic if and only if, $\rho<1$.

where $\rho=E(X)\left[1-R^*(\lambda)\right]+\lambda E(X)\left(E(S_0)\left(1+\alpha_0E(G_0)\right)+\sum_{k=1}^{m}r_kE(S_k)\left(1+\alpha_kE(G_k)\right)+pE(V)\right)..$

3. Steady state distribution

In this section, we first develop the steady state difference-differential equations for the retrial system by treating the elapsed retrial time, the elapsed service time, the elapsed vacation time and the elapsed repair time as supplementary variables. Then we derive the probability generating functions for the server state and the number of customers in the system/orbit.

The following probabilities are used in sequent sections:
$P_0(t)$ is the probability that the system is empty at time *t*. $P_n(x,t)$ is the probability that at time *t* there are exactly *n* customers in the orbit withthe elapsed retrial time of the test customers undergoing retrial is *x*. $\prod_{0,n}(x,t)$, is the probability that at time *t* there are exactly *n* customers in the orbit withthe elapsed service time on FES of the test customer undergoing service is *x*. $\prod_{k,n}(x,t)$,($1\leq k \leq m$) is the probability that at time *t* there are exactly *n* customers in theorbit with the elapsed service time on SOS of the test customer undergoing service is*x*. $\Omega_n(x,t)$ is the probability that at time *t* there are exactly *n* customers in the orbit with the elapsed vacation time is *x*. $R_{0,n}(x,y,t)$, is the probability that at time *t* there are exactly *n* customers in the orbit with the elapsed service time of the test customer undergoing service is *x* and the elapsed repair time on FES of server is *y*. $R_{k,n}(x,y,t)$, ($1\leq k \leq m$) is the probability that at time *t* there are exactly *n* customers in the orbit with the elapsed service time of the test customer undergoing service is *x* and the elapsed repair time on SOS of server is *y*.

For the process $\{N(t),\ t\geq 0\}$, we define the probabilities $P_0(t)=P\{C(t)=0,\ N(t)=0\}$ and the probability densities

$$P_n(x,t)dx = P\left\{C(t)=0, N(t)=n,\ x\leq R^0(t) < x+dx\right\},\ \text{for } t\geq 0,\ x\geq 0 \text{ and } n\geq 1,$$

$$\prod\nolimits_{0,n}(x,t)dx = P\left\{C(t)=1, N(t)=n,\ x\leq S_0{}^0(t) < x+dx\right\},\ \text{for } t\geq 0,\ x\geq 0 \text{ and } n\geq 0,$$

$$\prod\nolimits_{k,n}(x,t)dx = P\left\{C(t)=2,\ N(t)=n,\ x\leq S_k{}^0(t) < x+dx\right\},\ \text{for } t\geq 0,\ x\geq 0,\ (1\leq k\leq m) \text{ and } n\geq 0,$$

$$\Omega_n(x,t)dx = P\left\{C(t)=3,\ N(t)=n,\ \text{x}\leq V^0(t) < x+dx\right\},\ \text{for } t\geq 0,\ x\geq 0 \text{ and } n\geq 0,$$

$$R_{0,n}(x,y,t)dy = P\left\{C(t)=4,\ N(t)=n,\ \text{y}\leq G_0{}^0(t) < y+dy\big/S_0{}^0(t)=x\right\},\text{for } t\geq 0,\ (x,y)\geq 0 \text{ and } n\geq 0,$$

$$R_{k,n}(x,y,t)dy = P\left\{C(t)=5,\ N(t)=n,\ \text{y}\leq G_k{}^0(t) < y+dy\big/S_k{}^0(t)=x\right\},\text{for } t\geq 0,(x,y)\geq 0,(1\leq k\leq m) \text{ and } n\geq 0.$$

We assume that the stability condition is fulfilled in the sequel and so that we can set

$$P_0 = \lim_{t\to\infty} P_0(t), \text{ for } t \geq 0,$$

$$P_n(x) = \lim_{t\to\infty} \psi_n(x,t), \text{ for } t \geq 0, x \geq 0 \text{ and } n \geq 1,$$

$$\prod_{0,n}(x) = \lim_{t\to\infty} P_{0,n}(x,t), \text{ for } t \geq 0,\ x \geq 0 \text{ and } n \geq 0,$$

$$\prod_{k,n}(x) = \lim_{t\to\infty} P_{k,n}(x,t), \text{ for } t \geq 0,\ x \geq 0 \text{ and } n \geq 0,\ (1 \leq k \leq m)$$

$$\Omega_n(x) = \lim_{t\to\infty} \Omega_n(x,t), \text{ for } t \geq 0,\ x \geq 0 \text{ and } n \geq 0,$$

$$R_{0,n}(x,y) = \lim_{t\to\infty} R_{0,n}(x,y,t), \text{ for } t \geq 0,\ (x,y) \geq 0, \text{ and } n \geq 0,$$

$$R_{k,n}(x,y) = \lim_{t\to\infty} R_{k,n}(x,y,t), \text{ for } t \geq 0,\ (x,y) \geq 0, \text{ and } n \geq 0, (1 \leq k \leq m).$$

By the method of supplementary variable technique, we obtain the following system of equations that govern the dynamics of the system behaviour.

$$\lambda P_0 = \int_0^\infty \Omega_0(x)\gamma(x)dx \tag{1}$$

$$\frac{dP_n(x)}{dx} + [\lambda + a(x)]P_n(x) = 0,\ n \geq 1 \tag{2}$$

$$\frac{d\prod_{0,n}(x)}{dx} + [\lambda + \alpha_0 + \mu_0(x)]\prod_{0,n}(x) = \lambda\sum_{u=1}^{n}\chi_u \prod_{0,n-u}(x) + \int_0^\infty \xi_0(y)R_{0,n}(x,y)dy, n \geq 1, \tag{3}$$

$$\frac{d\prod_{k,n}(x)}{dx} + [\lambda + \alpha_k + \mu_k(x)]\prod_{k,n}(x) = \lambda\sum_{u=1}^{n}\chi_u \prod_{k,n-u}(x) + \int_0^\infty \xi_k(y)R_{k,n}(x,y)dy, n \geq 1, (1 \leq k \leq m) \tag{4}$$

$$\frac{d\Omega_0(x)}{dx} + [\lambda + \gamma(x)]\Omega_0(x) = 0 \tag{5}$$

$$\frac{d\Omega_n(x)}{dx} + [\lambda + \gamma(x)]\Omega_n(x) = \lambda\sum_{u=1}^{n}\chi_u\Omega_{n-u}(x), n \geq 1 \tag{6}$$

$$\frac{dR_{0,n}(x,y)}{dy} + [\lambda + \xi_0(y)]R_{0,n}(x,y) = \lambda\sum_{u=1}^{n}\chi_u R_{0,n-u}(x,y),\ n \geq 1 \tag{7}$$

$$\frac{dR_{k,n}(x,y)}{dy} + [\lambda + \xi_k(y)]R_{k,n}(x,y) = \lambda\sum_{u=1}^{n}\chi_u R_{k,n-u}(x,y),\ n \geq 1, (1 \leq k \leq m) \tag{8}$$

The steady state boundary conditions at $x = 0$ and $y = 0$ are

$$P_n(0) = qr_0\int_0^\infty \Pi_{0,n}(x)\mu_0(x)dx + q\sum_{k=1}^{m}\int_0^\infty \Pi_{k,n}(x)\mu_k(x)dx + \int_0^\infty \Omega_n(x)\gamma(x)dx,\ n\ge 1 \tag{9}$$

$$\Pi_{0,n}(0) = \int_0^\infty P_{n+1}(x)a(x)dx + \lambda\sum_{u=1}^{n}\chi_u\int_0^\infty P_{n-u+1}(x)dx + \lambda\chi_{n+1}P_0,\ n\ge 1 \tag{10}$$

$$\Pi_{k,n}(0) = r_k\int_0^\infty \Pi_{0,n}(x)\mu_0(x)dx,\ n\ge 1,\ (1\le k\le m) \tag{11}$$

$$\Omega_0(0) = \left\{ r_0\int_0^\infty \Pi_{0,0}(x)\mu_0(x)dx + \sum_{k=1}^{m}\int_0^\infty \Pi_{k,0}(x)\mu_k(x)dx \right\},\ n=0,\ (1\le k\le m) \tag{12}$$

$$\Omega_n(0) = p\left\{ r_0\int_0^\infty \Pi_{0,n}(x)\mu_0(x)dx + \sum_{k=1}^{m}\int_0^\infty \Pi_{k,n}(x)\mu_k(x)dx \right\},\ n=0,\ (1\le k\le m) \tag{13}$$

$$R_{0,n}(x,0) = \alpha_0\Pi_{0,n}(x),\ n\ge 0, \tag{14}$$

$$R_{k,n}(x,0) = \alpha_k\Pi_{k,n}(x),\ n\ge 0,\ \text{for}(1\le k\le m) \tag{15}$$

The normalizing condition is

$$\begin{aligned} P_0 + \sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx + \sum_{n=0}^{\infty}\left(\int_0^\infty P_{0,n}(x)dx + \sum_{k=1}^{m}\int_0^\infty P_{k,n}(x)dx + \int_0^\infty \Omega_n(x)dx \right. \\ \left. + \sum_{n=0}^{\infty}\left(\int_0^\infty\int_0^\infty R_{0,n}(x,y)dxdy + \sum_{k=1}^{m}\left[\int_0^\infty\int_0^\infty R_{k,n}(x,y)dxdy \right] \right) \right) = 1 \end{aligned} \tag{16}$$

To solve the above equations, then we define the generating functions for $|z| \le 1$, for $(1\le k\le m)$

$$P(x,z) = \sum_{n=1}^{\infty}P_n(x)z^n\ ;\ P(0,z) = \sum_{n=1}^{\infty}P_n(0)z^n;\ \Pi_0(x,z) = \sum_{n=0}^{\infty}\Pi_{0,n}(x)z^n\ ;\Pi_0(0,z) = \sum_{n=0}^{\infty}\Pi_{0,n}(0)z^n;$$

$$\Pi_k(x,z) = \sum_{n=0}^{\infty}\Pi_{k,n}(x)z^n;\ \Pi_k(0,z) = \sum_{n=0}^{\infty}\Pi_{k,n}(0)z^n;\ \Omega(x,z) = \sum_{n=1}^{\infty}\Omega_n(x)z^n\ ;\ \Omega(0,z) = \sum_{n=1}^{\infty}\Omega_n(0)z^n\ ;$$

$$R_0(x,y,z) = \sum_{n=0}^{\infty}R_{0,n}(x,y)z^n\ ;R_0(x,0,z) = \sum_{n=0}^{\infty}R_{0,n}(x,0)z^n\ ;\ R_k(x,y,z) = \sum_{n=0}^{\infty}R_{k,n}(x,y)z^n;$$

$$R_k(x,0,z) = \sum_{n=0}^{\infty}R_{k,n}(x,0)z^n;\ \text{and}\quad X(z) = \sum_{n=1}^{\infty}\chi_n z^n$$

Now multiplying the steady state equation and steady state boundary condition (1)–(15) by z^n and summing over n, (n = 0,1,2...and 1≤k≤m)

$$\frac{dP(x,z)}{dx}+[\lambda+a(x)]P(x,z)=0 \tag{17}$$

$$\frac{d\prod_0(x,z)}{dx}+[\lambda(1-X(z))+\alpha_0+\mu_0(x)]\prod_0(x,z)=\int_0^\infty \xi_0(y)R_0(x,y,z)dy \tag{18}$$

$$\frac{d\prod_k(x,z)}{dx}+[\lambda(1-X(z))+\alpha_k+\mu_k(x)]\prod_k(x,z)=\int_0^\infty \xi_k(y)R_k(x,y,z)dy \tag{19}$$

$$\frac{d\Omega(x,z)}{dx}+[\lambda(1-X(z))+\gamma(x)]\Omega(x,z)=0 \tag{20}$$

$$\frac{dR_0(x,y,z)}{dy}+[\lambda(1-X(z))+\xi_0(y)]R_0(x,y,z)=0 \tag{21}$$

$$\frac{dR_k(x,y,z)}{dy}+[\lambda(1-X(z))+\xi_k(y)]R_k(x,y,z)=0 \tag{22}$$

The steady state boundary conditions at $x = 0$ and $y = 0$ are

$$P(0,z)=q\left\{r_0\int_0^\infty \prod_0(x,z)\mu_0(x)dx+\sum_{k=1}^{m}\int_0^\infty \prod_k(x,z)\mu_k(x)dx\right\}+\int_0^\infty \Omega(x,z)\gamma(x)dx-\lambda bP_0-q\Omega_0(0) \tag{23}$$

$$\prod_0(0,z)=\frac{1}{z}\int_0^\infty P(x,z)a(x)dx+\frac{\lambda X(z)}{z}\int_0^\infty P(x,z)dx+\frac{\lambda X(z)}{z}P_0 \tag{24}$$

$$\prod_k(0,z)=r_k\int_0^\infty \prod_0(x,z)\mu_0(x)dx \tag{25}$$

$$\Omega(0,z)=p\left\{r_0\int_0^\infty \prod_0(x,z)\mu_0(x)dx+\sum_{k=1}^{m}\int_0^\infty \prod_k(x,z)\mu_k(x)dx\right\} \tag{26}$$

$$R_0(x,0,z)=\alpha_0P_0(x,z) \tag{27}$$

$$R_k(x,0,z)=\alpha_kP_k(x,z) \tag{28}$$

Solving the partial differential equations (17)-(22), it follows that for ($1\le k\le m$)

$$P(x,z)=P(0,z)[1-R(x)]\exp^{-\lambda x} \tag{29}$$

$$\prod_0(x,z)=\prod_0(0,z)[1-S_0(x)]\exp^{-A_0(z)x} \tag{30}$$

$$\prod_k(x,z)=\prod_k(0,z)[1-S_k(x)]\exp^{-A_k(z)x} \tag{31}$$

$$\Omega(x,z)=\Omega(0,z)[1-V(x)]\exp^{-b(z)x} \tag{32}$$

$$R_0(x,y,z)=R_0(x,0,z)[1-G_0(y)]\exp^{-b(z)y} \tag{33}$$

$$R_k(x,y,z)=R_k(x,0,z)[1-G_k(y)]\exp^{-b(z)y} \tag{34}$$

$$A_0(z)=b(z)+\alpha_0[1-G_0^*(b(z))],$$
$$A_k(z)=b(z)+\alpha_k[1-G_k^*(b(z))] \text{ and } b(z)=\lambda\left(1-X(z)\right)$$

From (5) we obtain,

$$\Omega_{j,0}(x)=\Omega_{j,0}(0)[1-V(x)]\exp^{-\lambda x} \tag{35}$$

Multiplying with equation (35) by $\gamma(x)$ on both sides and integrating with respect to $x$from 0 to ∞, then from (1) we have,

$$\Omega_0(0)=\frac{\lambda P_0}{V^*(\lambda)} \tag{36}$$

Inserting (29) in (24), we obtain

$$\Pi_0(0,z)=\frac{P(0,z)}{z}\left[R^*(\lambda)+X(z)(1-R^*(\lambda))\right]+\frac{\lambda X(z)}{z}P_0 \tag{37}$$

Inserting (30) in (25), we obtain

$$\Pi_k(0,z)=r_k\,\Pi_0(0,z)\left[S_0^*(A_0(z))\right] \tag{38}$$

Inserting (32) in (26), we obtain

$$\Omega(0,z)=\frac{q\lambda P_0}{V^*(\lambda)}+p\Pi_0(0,z)S_0^*[A_0(z)]\left(r_0+\sum_{k=1}^{m}r_kS_k^*[A_k(z)]\right) \tag{39}$$

Inserting (33)-(34) in (27)-(28), we obtain

$$R_0(x,0,z)=\alpha_0\,\Pi_0(0,z)[1-S_0(x)]\exp^{-A_0(z)x} \tag{40}$$

$$R_k(x,0,z)=\alpha_k\,\Pi_k(0,z)[1-S_k(x)]\exp^{-A_k(z)x} \tag{41}$$

Using (23) and (37)-(39), finally we get,

$$P(0,z)=q\Pi_0(0,z)S_0^*(A_0(z))\left(r_0+\sum_{k=1}^{m}r_k\left[S_k^*(A_k(z))\right]\right)+\Omega(0,z)V^*[b(z)]-\lambda P_0-\frac{q\lambda P_0}{V^*(\lambda)} \tag{42}$$

Solving (42),

$$P(0,z)=\frac{\lambda P_0}{V^*(\lambda)}\times\frac{Nr(z)}{Dr(z)} \tag{43}$$

$$Nr(z)=V^*(\lambda)\left\{X(z)S_0^*[A_0(z)]\left[r_0+\sum_{k=1}^{m}r_kS_k^*[A_k(z)]\right]\left(q+pV^*\left[b(z)\right]\right)-z\right\}+zq\left(V^*\left[b(z)\right]-1\right)$$

$$Dr(z)=z-\left[R^*(\lambda)+X(z)(1-R^*(\lambda))\right]\left(S_0^*[A_0(z)]\left[r_0+\sum_{k=1}^{m}r_kS_k^*[A_k(z)]\right]\left(q+pV^*\left[b(z)\right]\right)\right)$$

Using (43) in (37), we get

$$\prod\nolimits_0(0,z)=\frac{\lambda P_0}{V^*(\lambda)}\times\frac{Nr(z)}{Dr(z)} \tag{44}$$

$$Nr(z)=\left\{\left(q\left(V^*\left[b(z)\right]-1\right)-V^*(\lambda b)\right)\left[R^*(\lambda)+X(z)(1-R^*(\lambda))\right]+V^*(\lambda)X(z)\right\}$$

Using (44) in (38), we get

$$\prod\nolimits_k(0,z)=\frac{r_k\lambda P_0}{V^*(\lambda)}\times\frac{Nr(z)}{Dr(z)} \tag{45}$$

$$Nr(z)=S_0^*[A_0(z)]\left\{\left(q\left(V^*\left[b(z)\right]-1\right)-V^*(\lambda b)\right)\left[R^*(\lambda)+X(z)(1-R^*(\lambda))\right]+V^*(\lambda)X(z)\right\}$$

Using (44) and (39), we get

$$\Omega(0,z)=\frac{\lambda P_0}{V^*(\lambda)}\times\frac{Nr(z)}{Dr(z)} \tag{46}$$

$$Nr(z)=qz-\left\{\left(q+pV^*(\lambda)\right)\left[R^*(\lambda)+X(z)(1-R^*(\lambda))\right]-pV^*(\lambda)X(z)\right\}\left(S_0^*[A_0(z)]\left[r_0+\sum_{k=1}^{m}r_kS_k^*[A_k(z)]\right]\right)$$

Using (44)-(45) in (40)-(41), we get

$$R_0(x,0,z)=\frac{\lambda P_0}{V^*(\lambda)}\times\frac{Nr(z)}{Dr(z)} \tag{47}$$

$$Nr(z)=\alpha_0[1-S_0(x)]\exp^{-A_0(z)x}\left\{V^*(\lambda)X(z)+\left(q\left(V^*\left[b(z)\right]-1\right)-V^*(\lambda)\right)\left[R^*(\lambda)+X(z)(1-R^*(\lambda))\right]\right\}$$

$$R_k(x,0,z)=\frac{\lambda P_0}{V^*(\lambda)}\times\frac{Nr(z)}{Dr(z)} \tag{48}$$

$$Nr(z)=\alpha_kr_kS_0^*[A_0(z)][1-S_k(x)]\exp^{-A_k(z)x}\left\{V^*(\lambda)X(z)+\left(q\left(V^*\left[b(z)\right]-1\right)-V^*(\lambda)\right)\left[R^*(\lambda)+X(z)(1-R^*(\lambda))\right]\right\}$$

Using (42)-(48) and (29)-(34), then we get the limiting probability generating functions $P(x,z),\prod_0(x,z),\prod_k(x,z),\ \Omega(x,z),\ R_0(x,y,z),R_k(x,y,z).$ Next we are interested in investigating the marginal orbit size distributions due to system state of the server.

Theorem 3.2.*Under the stability condition ρ < 1, the stationary distributions of the number of customers in the system when server being idle, busy on both phases, on vacation and under repair on both phases (for*$1\leq k\leq m$*) are given by*

$$P(z)=\frac{P_0[1-R^*(\lambda)]}{V^*(\lambda)}\times\frac{Nr(z)}{Dr(z)} \tag{49}$$

$$Nr(z)=V^*(\lambda)\left\{X(z)S_0^*[A_0(z)]\left[r_0+\sum_{k=1}^{m}r_kS_k^*[A_k(z)]\right]\Big(q+pV^*\big[b(z)\big]\Big)-z\right\}+zq\Big(V^*\big[b(z)\big]-1\Big)$$

$$Dr(z)=z-\Big[R^*(\lambda)+X(z)(1-R^*(\lambda))\Big]\left(S_0^*[A_0(z)]\left[r_0+\sum_{k=1}^{m}r_kS_k^*[A_k(z)]\right]\Big(q+pV^*\big[b(z)\big]\Big)\right)$$

$$\Pi_0(z)=\frac{\lambda P_0\Big(1-S_0^*(A_0(z))\Big)}{A_0(z)V^*(\lambda)Dr(z)}\Big\{V^*(\lambda)X(z)+\Big(q\Big(V^*\big[b(z)\big]-1\Big)-V^*(\lambda b)\Big)\Big[R^*(\lambda)+X(z)(1-R^*(\lambda))\Big]\Big\} \tag{50}$$

$$\Pi_k(z)=\frac{r_k\lambda P_0S_0^*[A_0(z)]\Big(1-S_k^*(A_k(z))\Big)}{A_k(z)V^*(\lambda)Dr(z)} \\ \times\Big\{V^*(\lambda)X(z)+\Big(q\Big(V^*\big[b(z)\big]-1\Big)-V^*(\lambda)\Big)\Big[R^*(\lambda)+X(z)(1-R^*(\lambda))\Big]\Big\} \tag{51}$$

$$\Omega(z)=\frac{\lambda P_0\Big(1-V^*[b(z)]\Big)}{b(z)V^*(\lambda)Dr(z)}\Bigg(S_0^*[A_0(z)]\left[r_0+\sum_{k=1}^{m}r_kS_k^*[A_k(z)]\right]\Bigg)\Bigg\} \\ \times\Big\{qz-\Big\{\Big(q+pV^*(\lambda b)\Big)\Big[R^*(\lambda)+X(z)(1-R^*(\lambda))\Big]-pV^*(\lambda)X(z)\Big\}$$

(52)

$$R_0(z)=\frac{\alpha_0\lambda P_0\Big(1-S_0^*(A_0(z))\Big)\Big(1-G_0^*(b(z))\Big)}{A_0(z)b(z)V^*(\lambda)Dr(z)} \\ \times\Big\{V^*(\lambda)X(z)+\Big(q\Big(V^*\big[b(z)\big]-1\Big)-V^*(\lambda)\Big)\Big[R^*(\lambda)+X(z)(1-R^*(\lambda))\Big]\Big\} \tag{53}$$

$$R_k(z)=\frac{r_k\alpha_k\lambda P_0S_0^*[A_0(z)]\Big(1-S_k^*(A_k(z))\Big)\Big(1-G_k^*(b(z))\Big)}{A_k(z)b(z)V^*(\lambda)Dr(z)} \\ \times\Big\{V^*(\lambda)X(z)+\Big(q\Big(V^*\big[b(z)\big]-1\Big)-V^*(\lambda)\Big)\Big[R^*(\lambda)+X(z)(1-R^*(\lambda))\Big]\Big\} \tag{54}$$

$$\text{Where } P_0=\frac{V^*(\lambda)\Big(1-E(X)(1-R^*(\lambda))-\varpi\Big)}{Dr=\Big(q\lambda E(V)+V^*(\lambda b)\Big)-V^*(\lambda)(1-R^*(\lambda))}; \tag{55}$$

$A_0(z)=b(z)+\alpha_0[1-G_0^*(b(z))]$,

$A_k(z)=b(z)+\alpha_k[1-G_k^*(b(z))]$ and $b(z)=\lambda\left(1-X(z)\right)$

Proof Integrating the above (29) - (32) equations with respect to x and define the partial probability generating functions as, for ($1 \le k \le m$)

$$P(z)=\int_0^\infty P(x,z)dx,\ \prod_0(z)=\int_0^\infty \prod_0(x,z)dx,\ \prod_k(z)=\int_0^\infty \prod_k(x,z)dx,\ \Omega(z)=\int_0^\infty \Omega(x,z)dx.$$

Integrating the above (33) - (34) equations with respect to x and y define the partial probability generating functions as, for (i = 1,2)

$$R_0(x,z)=\int_0^\infty R_0(x,y,z)dy,\ R_0(z)=\int_0^\infty R_0(x,z)dx,\ R_k(x,z)=\int_0^\infty R_k(x,y,z)dy,\ R_k(z)=\int_0^\infty R_k(x,z)dx.$$

Since, the only unknown is P_0 the probability that the server is idle when no customer in the orbit and it can be determined using the normalizing condition($1 \le k \le m$). Thus, by setting z = 1 in (49) – (54) and applying L – Hospitals rule whenever necessary and we get $P_0+P(1)+\Omega(1)+\prod_0(1)+R_0(1)+\sum_{k=1}^{m}\left(\prod_k(1)+R_k(1)\right)=1.$

Theorem3.4. *Under the stability condition ρ < 1,*

The probability generating function of the number of customer in the system ($K(z)$) is obtained by using

$$K(z)=P_0+P(z)+\Omega(z)+z\left[\prod_0(z)+R_0(z)+\sum_{k=1}^{m}\left(\prod_k(z)+R_k(z)\right)\right] \tag{56}$$

The probability generating function of the number of customer in the orbit ($H(z)$) is obtained by using

$$H(z)=P_0+P(z)+\Omega(z)+\prod_0(z)+R_0(z)+\sum_{k=1}^{m}\left(\prod_k(z)+R_k(z)\right) \tag{57}$$

Substituting (49) – (55) in the above results (56) and (57) can be obtained by direct calculation.

4. Performance Measures

In this section, we obtain the mean system size, mean orbit size, mean waiting time in the system and orbit which are as follows,

Theorem 4.1. Let L_s, L_q, W_s and W_q be the mean number of customers in the system, the mean number of customers in the orbit, average time a customer spends in the system and average time a customer spends in the orbit using Little's formula respectively, then under the stability condition, we have

$$L_q = \frac{P_0}{V^*(\lambda)}\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

$$Nr_q''(1) = -2E(X)\left(V^*(\lambda)R^*(\lambda) + q\lambda E(V)\right)$$

$$\begin{aligned} Nr_q'''(1) = {} & 3\left\{-(w_0 + w_k)\left(V^*(\lambda)(1 - R^*(\lambda))\right) + w_k\left\{-\left(q + pV^*(\lambda)\right)(1 - R^*(\lambda)) - q\right\}\right. \\ & + \omega V^*(\lambda)\left\{2V^*(\lambda b)(1 - R^*(\lambda))\right\} + V^*(\lambda)(1 - R^*(\lambda))\left\{E(X(X-1)) + E(X)\right\} \\ & \left. + \lambda E(X)E(V)\left\{pV^*(\lambda)(1 - R^*(\lambda))\left\{2\omega E(X) + E(X(X-1))R^*(\lambda)\right\}\right.\right\} \end{aligned}$$

$$Dr_q''(1) = -2E(X)\left(1 - (1 - R^*(\lambda))E(X) - \varpi\right)$$

$$Dr_q''' = -3\left\{\begin{array}{l} \left\{E(X)\left(\left(w_0 + w_k + 2\delta + \omega\lambda E(X)E(V)\right) + pw_v + \varpi E(X) + E(X(X-1))\right)\right\} \\ + E(X(X-1))\left(1 - (1 - R^*(\lambda))E(X) - \varpi\right) \end{array}\right\}$$

$$L_s = \frac{P_0}{V^*(\lambda)}\left[\frac{Nr_s'''(1)Dr_q''(1) - Dr_q'''(1)Nr_s''(1)}{3\left(Dr_q''(1)\right)^2}\right]$$

$$Nr_s'''(1) = Nr_q'''(1) + 6\omega E(X)\left(V^*(\lambda)R^*(\lambda) + q\lambda E(V)\right)$$

where

$$\delta = \left(\lambda E(X)\right)^2\left(\sum_{k=1}^{m} r_k E(S_0)E(S_k)\left(1 + \alpha_0 E(G_0)\right)\left(1 + \alpha_k E(G_k)\right)\right)$$

$$\omega = \lambda E(X)\left(E(S_0)\left(1 + \alpha_0 E(G_0)\right)\right) + \sum_{k=1}^{m} r_k E(S_k)\left(1 + \alpha_k E(G_k)\right)\right)$$

$$w_0 = \left(\lambda E(X)\right)^2 E(S_0^2)\left(1 + \alpha_0 E(G_0)\right)\right)^2 + \lambda E(S_0)E(X(X-1))\left(1 + \alpha_0 E(G_0)\right)\right) + \alpha_0 E(S_0)\left(\lambda E(X)\right)^2 E(G_0^2)$$

$$w_k = \sum_{k=1}^{m} r_k\left\{\begin{array}{l} \left(\lambda E(X)\right)^2 E(S_k^2)\left(1 + \alpha_k E(G_k)\right)^2 + \lambda E(S_k)E(X(X-1)) \\ \left(1 + \alpha_k E(G_k)\right) + \alpha_k E(S_k)E(G_k^2)\left(\lambda E(X)\right)^2 \end{array}\right\}$$

$$w_v = \left(\lambda E(X)\right)^2 E(V^2) + \lambda E(X(X-1)))E(V)$$

Proof. The mean number of customers in the orbit (L_q) under steady state condition is obtained by differentiating (57) with respect to z and evaluating at $z = 1$

$$L_q = \frac{Nr(z)}{Dr(z)} = H'(1) = \lim_{z\to 1}\frac{d}{dz}H(z).$$

The mean number of customers in the system (L_s) under steady state condition is obtained by differentiating (56) with respect to z and evaluating at $z = 1$

$$L_s = \frac{Nr(z)}{Dr(z)} = K'(1) = \lim_{z\to 1}\frac{d}{dz}K(z)\ .$$

The average time a customer spends in the system (W_s) and orbit (W_q) understeady-state condition due to Little's formula is,

$$W_S = \frac{L_S}{\lambda E(X)}$$

The average time a customer spends in the and orbit (W_q) under steady- state condition due to Little's formula is,

$$W_q = \frac{L_q}{\lambda E(X)}$$

The stated formula follows by direct calculation.

5. Special cases

In this section, we discuss some special cases of our model.

Case 1: Let us consider Single phase and No breakdown, then this model can be reduced to an M/G/1 retrial queue with general retrial time under Bernoulli vacations.

$$K(z) = \frac{P_0\left\{\left(z+(1-z)R^*(\lambda)\right)\left[1-V^*\left(\lambda-\lambda z\right)\right]+(1-z)V^*(\lambda)R^*(\lambda)\ \right\}S_0^*\left[\lambda-\lambda z\right]}{\left\{\left[R^*(\lambda)+z(1-R^*(\lambda))\right]S_0^*\left[\lambda-\lambda z\right]-z\right\}}$$

where $P_0 = \dfrac{\left[R^*(\lambda)\ -\lambda E(S_0)\right]}{\left[\lambda E(V)+V^*(\lambda)R^*(\lambda)\ \right]}$

The above results are equivalent the results obtained by Krishnakumar and Arivudainambi (2002)

Case 2:Let us consider Single phase, No vacation and No breakdown, then our model can be reduced to M/G/1 retrial queue. The following form and results agree with Gomez-Corral (1999).

$$K(z)=\left\{\frac{\left[R^{*}(\lambda)-\lambda E(S_0)\right]S_0^{*}\left[\lambda-\lambda z\right]\left[z-1\right]}{z-\left[R^{*}(\lambda)+z(1-R^{*}(\lambda))\right]\left\{S_0^{*}\left[\lambda-\lambda z\right]\right\}}\right\};\ L_q=\frac{\left\{\lambda^2E(S_0{}^2)+2\lambda E(S_0)\left(1-R^{*}(\lambda)\right)\right\}}{2\left\{R^{*}(\lambda)-\lambda E(S_0)\right\}}$$

Case 3:Letus consider Single phase, No feedback, No balking and reneging, No vacation and No breakdown, then our model can be reduced to M/G/1 queue.

$$K(z)=\left\{\frac{\left[1-\lambda E(S_0)\right]S_0^{*}\left[\lambda-\lambda z\right]\left[z-1\right]}{z-S_0^{*}\left[\lambda-\lambda z\right]}\right\};\quad L_q=\frac{\lambda^2E(S_0{}^2)}{2\left[1-\lambda E(S_0)\right]}$$

6. Numerical examples

In this section, we discussed some numerical examples in order to illustrate the effect of various parameters in the system performance measures of our system where all retrial times, service times, vacation times and repair times are exponentially, Erlangianly and hyper-exponentially distributed. We assume arbitrary values to the parameters such that the steady state condition is satisfied.

From the following figures, we can visualize the effect of $\bar{\alpha}$and $p$on the system performance measures. Figure-1 shows that the mean orbit size L_q is increasing for the increasing the values of the failure probability$\bar{\alpha}$. Figure-2 shows that the idle probability P_0 increases for the increasing values of the vacation probability p.

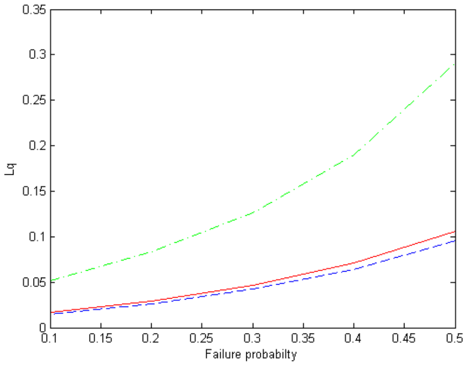

Figure 1: L_q versus $\bar{\alpha}$

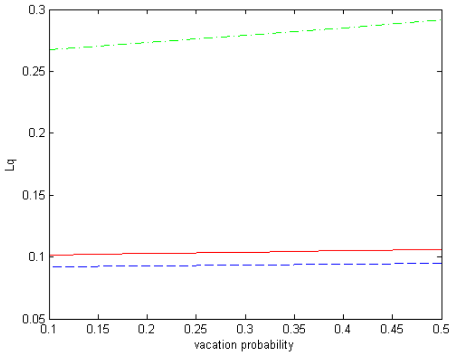

Figure 2: P_0 versus p

6. Conclusion

In this work, we have discussed a batch arrival retrial queue with second optional services, modified extended Bernoulli vacations where subject to server breakdowns and repair. The probability generating functions of the number of customers in the system and orbit are found by using the supplementary variable technique. The performance measures like, the mean number of customers in the system/orbit, the average waiting time of customer in the system/orbit are obtained. The suggested model has potential application in the transfer model of an email sys-tem. Simple mail transfer protocol (SMTP) is used to deliver the messages between the mail servers.

REFERENCES:

[1] Artalejo, J. R. (1999). Accessible bibliography on retrial queues, Mathematical and Computer Modelling, 30, 1-6.

[2] Artalejo, J. R. and Choudhury, G. (2004). Steady state analysis of an M/G/1 queue with repeated attempts and two-phase service. Quality Technology & Quantitative Management, 1(2), 189-199.

[3] Choudhury, G., Tadj, L. and Deka, K. (2010), A Batch arrival retrial queueing system with two phases of service and service interruption, Computers and Mathematics with Applications 59, 437- 450.

[4] Choudhury, G. and Deka, K. (2008). An M/G/1 retrial queueing system with two phases of service subject to the server breakdown and repair. Performance Eval. 65, 714-724.

[5] Doshi, B. (1986). Queueing systems with vacations-a survey, Queueing Systems, 1, 29-66.

[6] Falin, G. I. and Templeton, J. C. G. (1997). Retrial Queues. Chapman & Hall, London.

[7] Gomez-Corral, A. (1999). Stochastic analysis of single server retrial queue with the general retrial times. Naval Research Logistics, 46, 561-581.

[8] Ke, J. C. and Chang, F. M. (2009), $M^{[X]}/(G1,G2)/1$ retrial queue under Bernoulli vacation schedule with general repeated attempts and starting failures, Applied Mathematical Modelling 33, 3186–3196.

[9] Ke, J. C. (2007). Operating characteristic analysis on the $M^{[X]}/G/1$ system with a variant vacation policy and balking, Applied Mathematical Modelling, 31: 1321–1337

[10] Krishnakumar, B. and Arivudainambi, D. (2002). The M/G/1 retrial queue with Bernoulli schedules and general retrial times. Comp. Math. Appl., 43, 15-30.

[11] Wang, J. and Li, J. (2009). A single server retrial queue with general retrial times and two phase service. JrlSystSci& Complexity, 22, 291-302.