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Information Density of Isospectral Potential

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Abstract— **The Quantum-mechanical information densities in position and momentum space measure the compactness in terms of the amount of information. The information density concepts of isospectral hydrogen potential are studied and their properties are analyzed. The momentum space information densities for the potential are obtained using Fourier transforms of the corresponding position space wave function. It is found that the momentum probability density function of the isospectral potential exhibits sharp distant peaks. The position space entropy densities have quite asymmetric shape depending on the values of quantum numbers. The position density function also possesses different behavior for various values of the parameter.**

***Index Terms*—Information Density, Isospectral Hamiltonians**

I. INTRODUCTION

INFORMATION entropy is an important quantity employed to study quantum mechanical systems. It provides a measure of information about the probability distribution in position and momentum space [1-11]. The position and momentum space entropies are given by the expressions

$$S_{pos} = -\int \rho(r) \ln \rho(r) dr \tag{1}$$

$$S_{mom} = -\int \rho(p) \ln \rho(p) dp \tag{2}$$

where $\rho(r) = \psi^*(r)\psi(r)$ and $\rho(p) = \psi^*(p)\psi(p)$ are the probability densities in position space and momentum space respectively and $\psi(p)$ is the Fourier transform of $\psi(r)$ in the momentum space.

We use the isospectral Hamiltonian approach to study the isospectral wave functions and their information densities. Two Hamiltonians are said to be strictly isospectral, if they have exactly same energy eigenvalue spectrum and S-matrix [12-14], whereas the wave functions and their dependent quantities are different. Though the idea of generating isospectral Hamiltonians using the Gelfand-Levitan approach or the Darboux procedures were known for some time, the supersymmetric quantum mechanical techniques make the procedure look simpler [15-17]. When one deletes a bound state of a given potential $V(x)$ and re-introduce the state, it involves solving a first order differential equation, which admits a free parameter. Thus, a set of one-dimensional family of potentials $\hat{V}(x,\lambda)$ can be constructed which have the exactly same energy spectrum as that of $V(x)$. In general, for any one dimensional potential with n bound states, one can construct an n-parameter family of strictly isospectral potentials, i.e. potentials with eigenvalues, reflection and transmission coefficients identical to those for original potential [17]. This aspect has been utilized profitably in many physical situations, which are of interest to various fields [18-20].

In this paper, we consider the Hydrogen potential and calculate the position and momentum space information density exactly for ground state and excited states of the isospectral potential. In section II, the isospectral Hamiltonian approach is briefly discussed. The deformed potential and their wave functions are constructed and used to calculate the information density for the isospectral potential in section III. In last section, we conclude with brief discussion.

II. ISOSPECTRAL HAMILTONIAN APPROACH

The relation between the bound state wave functions and the potential is utilized in solving exactly for the spectrum of one-dimensional potential problems. Once, the ground state wave function (ψ_0) is obtained and choose its energy to be zero, we can factorize the Hamiltonian [17] as $H_1 = A^\dagger A$ where $A = \frac{d}{dx} + W(x)$ and $A^\dagger = -\frac{d}{dx} + W(x)$ (in units $\hbar = 2m = 1$) are the supersymmetric operators and $W(x) = -\frac{d}{dx}[\ln \psi_0(x)]$ is the superpotential. We have

$$H_1\psi_n = A^\dagger A\psi_n = \varepsilon_n\psi_n, \tag{3}$$

Multiplying both sides with *A*, we obtain

$$AA^\dagger(A\psi_n) = \varepsilon_n(A\psi_n),$$

$$H_2(A\psi_n) = \varepsilon_n(A\psi_n). \tag{4}$$

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H_1 and H_2 are the supersymmetric partner Hamiltonian with eigenfunctions $\chi_n = A\psi_n$. H_2 has the same eigenvalue spectrum as that of H_1, but for the case $A\psi_0 = 0$, which is the case of supersymmetry broken. The relation between Hamiltonians reads,

$$E_n^{(2)} = E_{n+1}^{(1)}; \quad E_0^{(1)} = 0,$$

$$\psi_n^{(2)} = [E_{n+1}^{(1)}]^{-\frac{1}{2}} A\psi_{n+1}^{(1)},$$

$$\psi_{n+1}^{(1)} = [E_n^{(2)}]^{-\frac{1}{2}} A^{\dagger}\psi_n^{(2)},$$

The supersymmetric partner potentials $V_1(x)$ and $V_2(x)$ are related through superpotential as

$$V_{1,2}(x) = W^2(x) \mp \frac{dW}{dx}. \tag{5}$$

Now, for the potential $V_2(x)$, the original potential $V_1(x)$ is not unique [15,16]. The argument is as follows. Suppose H_2 has another factorization $BB^{\dagger}$, where $B = \frac{d}{dx} + \hat{W}(x)$, then, $H_2 = AA^{\dagger} = BB^{\dagger}$ but $H_1 = B^{\dagger}B$ is not $A^{\dagger}A$ rather it defines a certain new Hamiltonian. For superpotential $\hat{W}(x)$, the partner potential $V_2(x)$ is

$$V_2(x) = \hat{W}^2(x) + \hat{W}'(x). \tag{6}$$

Consider the most general solution as $\hat{W}(x) = W(x) + \phi(x)$, which demands that,

$$\phi^2(x) + 2W(x)\phi(x) + \phi'(x) = 0. \tag{7}$$

The solution of the above equation is $\phi(x) = \frac{d}{dx}\ln[I(x) + \lambda]$, where $I(x) = \int_{-\infty}^{x} \psi_0^2(x')dx'$ and λ is a constant. Therefore, we obtain,

$$\hat{W}(x) = W(x) + \frac{d}{dx}\ln[I(x) + \lambda]. \tag{8}$$

The corresponding one-parameter family of potentials $\hat{V}_1(x,\lambda)$ is given as

$$\hat{V}_1(x,\lambda) = V_1(x) - 2\frac{d^2}{dx^2}(\ln(I(x) + \lambda)). \tag{9}$$

The normalized ground state wave function corresponding to the potential $\hat{V}_1(x,\lambda)$ reads

$$\hat{\psi}_0(x,\lambda) = \frac{\sqrt{\lambda(1+\lambda)}\psi_0(x)}{I(x) + \lambda}, \tag{10}$$

where $\lambda \notin (0,-1)$. The excited state eigenfunctions for the potential $\hat{V}_1(x,\lambda)$ are given by [17],

$$\hat{\psi}_{n+1}(x,\lambda) = \psi_{n+1}(x) + \frac{1}{E_{n+1}}\left(\frac{I'(x)}{I(x) + \lambda}\right) \times \left(\frac{d}{dx} + W(x)\right)\psi_{n+1}(x). \tag{11}$$

Using the similar procedure, the two-parameter ground state wave function can be obtained as

$$\hat{\hat{\psi}}_0(x,\lambda_0,\lambda_1) = \frac{1}{\phi_0(x,\lambda_0,\lambda_1)} = \frac{\psi_0(x)}{\hat{A}_1(\lambda_1)A_1(I_0(x) + \lambda_0)}. \tag{12}$$

The equations (9), (10) and (11) represent the one-parameter family of isospectral potentials and wave functions, which shall be used to calculate the information density.

III. INFORMATION DENSITY OF ISOSPECTRAL HYDROGEN POTENTIAL

The one dimensional hydrogen atom is an interesting mathematical and physical problem to study bound state. The Hydrogen atom is described by the Coulomb potential,

$$V(x) = -\frac{1}{|x|} \tag{13}$$

The ground state wave function is given by [21, 22],

$$\psi(x) = \alpha^{-\frac{1}{2}} e^{-\frac{|x|}{\alpha}} \tag{14}$$

For excited states

$$\psi_{even}(x) = \sqrt{\frac{2}{n^5}}\, e^{-\frac{|x|}{n}}\, |x|\, L_{n-1}^{1}\left(\frac{2|x|}{n}\right)$$

$$\psi_{odd}(x) = \sqrt{\frac{2}{n^5}}\, e^{-\frac{|x|}{n}}\, x\, \mathrm{L}^1_{n-1}\left(\frac{2|x|}{n}\right) \quad (15)$$

The corresponding eigenfunctions in momentum space are

$$\psi_0(p) = \sqrt{\frac{2}{\pi}}\, \frac{\alpha^{\frac{1}{2}}}{(1+p^2\alpha^2)} \quad (16)$$

$$\psi_n(p) = \sqrt{\frac{2n}{\pi}}\, \frac{e^{\pm 2 i n \tan^{-1}(n p)}}{(1+p^2 n^2)} \quad (17)$$

Using isospectral Hamiltonian approach, the ground state wave function is obtained as

$$\psi_0(x,\lambda) = \frac{2\sqrt{\lambda(\lambda+1)}}{\sqrt{\alpha}}\, \frac{e^{-\frac{|x|}{\alpha}}}{2(\lambda+1) - e^{-\frac{2|x|}{\alpha}}} \quad (18)$$

After some calculation, the excited state isospectral wave function for odd values of n is calculated as

$$\psi_{n+1}(x,\lambda) = \sqrt{\frac{2}{n^5}}\, e^{-\frac{|x|}{n}} \left[x\, \mathrm{L}^1_{n-1}\left(\frac{2|x|}{n}\right) \left\{ 1 - \frac{e^{-\frac{2|x|}{n}}}{(n+1)^2\alpha^2 \left(\lambda+1-\frac{1}{2}e^{-\frac{2|x|}{n}} \right)} \right\} - \frac{e^{-\frac{2|x|}{n}}}{(n+1)^2\alpha \left(\lambda+1-\frac{1}{2}e^{-\frac{|x|}{n}} \right)} \left(1-\frac{x}{n}\right) \mathrm{L}^1_{n-1}\left(\frac{2|x|}{n}\right) - \frac{2x}{n}\, \mathrm{L}^2_{n-2}\left(\frac{2|x|}{n}\right) \right] \quad (19)$$

and for even n, we have

$$\psi_{n+1}(x,\lambda) = \sqrt{\frac{2}{n^5}}\, e^{-\frac{|x|}{n}} \left[|x|\, \mathrm{L}^1_{n-1}\left(\frac{2|x|}{n}\right) \left\{ 1 - \frac{e^{-\frac{2|x|}{n}}}{(n+1)^2\alpha^2 \left(\lambda+1-\frac{1}{2}e^{-\frac{2|x|}{n}} \right)} \right\} - \frac{e^{-\frac{2|x|}{n}}}{(n+1)^2\alpha \left(\lambda+1-\frac{1}{2}e^{-\frac{|x|}{n}} \right)} \left\{ \left(1-\frac{|x|}{n}\right) \mathrm{L}^1_{n-1}\left(\frac{2|x|}{n}\right) - \frac{2|x|}{n}\, \mathrm{L}^2_{n-2}\left(\frac{2|x|}{n}\right) \right\} \right] \quad 20)$$

In momentum space, the ground state isospectral wave function is calculated after some lengthy but straight forward calculations as

$$\psi_0(p,\lambda) = \sqrt{72\,\pi\,\alpha\,\lambda(\lambda+1)}\, \frac{1}{[3\pi(2\lambda+1)(1+\alpha^2p^2)+t_1]} \quad (21)$$

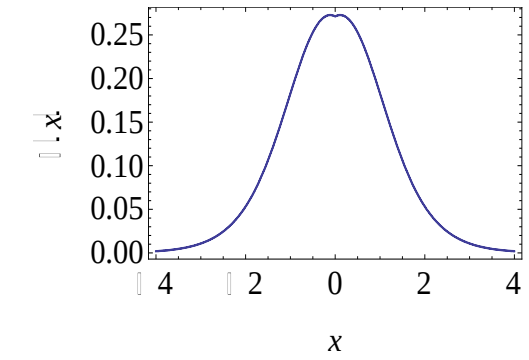
$$t_1 = 6(1+\alpha^2p^2)\tan^{-1}(\alpha\, p) + p$$

The excited state isospectral wave function in momentum space reads,

$$\psi_n(p,\lambda) = \frac{Exp[2\, i\, n\, \tan^{-1}(n\, p)]\, \sqrt{\frac{2}{\pi}}\, [n^2\,(1+n^2p^2)^3 + t_2]}{n^{\frac{3}{2}}\,(1+n^2p^2)^4} \quad (22)$$

$$t_2 = \frac{4\,\alpha\left(-p\,\alpha^2+n^2\left(p-i\left(1+\alpha^2p^2\right)\right)\right)}{(1+\alpha^2p^2)\,(\pi+\lambda\,\pi+2\tan^{-1}(n\,p))}$$

Using isospectral wave functions, the information density is calculated for different states and their characteristic features are graphically demonstrated for some levels of the potential. The figures 1 and 2 demonstrates the variation of information density as a function of deformation parameter in position and momentum space respectively. In figures 3 and 4, the information density is analyzed with the variation in parameter **α** in addition to the deformation parameter.



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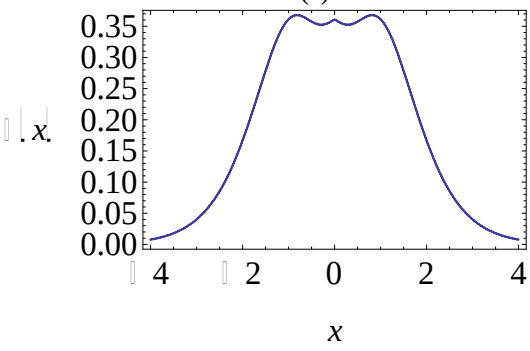

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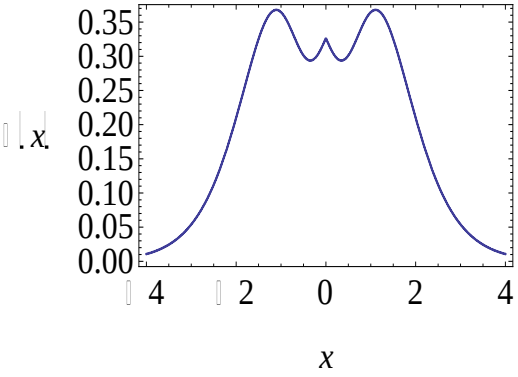

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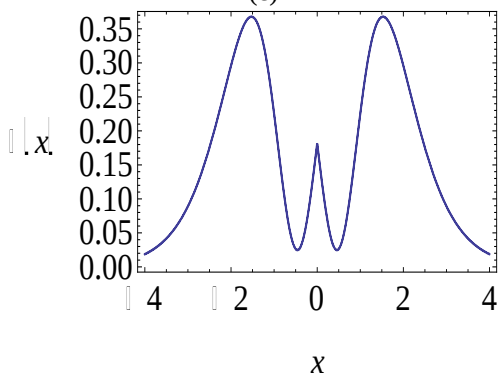
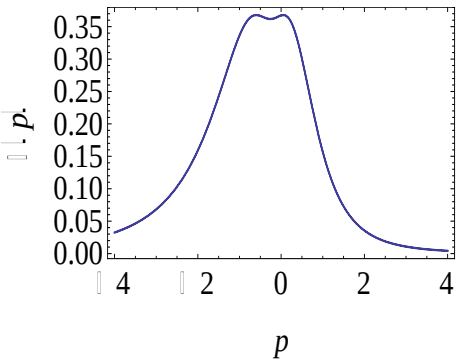


(d)

Fig 1: Information density in position space for isospectral Hydrogen potential with **α** =1 and deformation parameter (a) c = 0.1, (b) c =0.22, (c) c=0.25, (d) c=0.3



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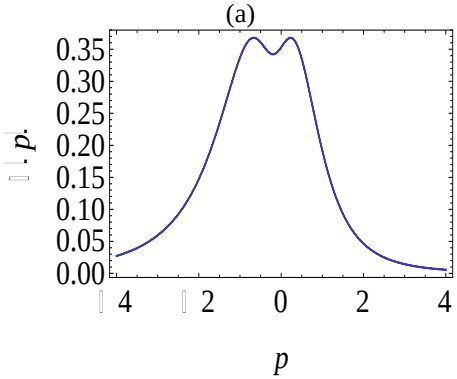

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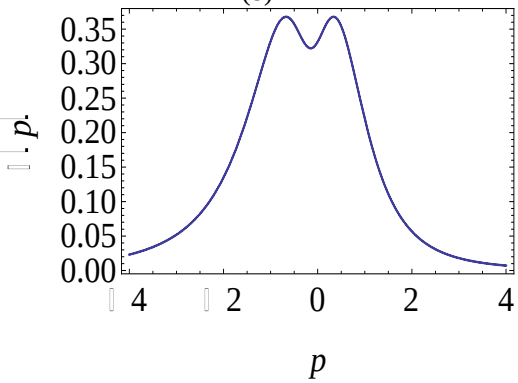

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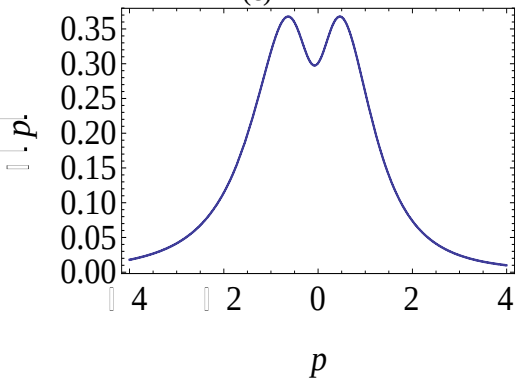

(d)

Fig 2: Information density in momentum space for isospectral Hydrogen potential with **α** =1 and deformation parameter (a) c = 0.3, (b) c =0.5, (c) c=0.8, (d) c=2

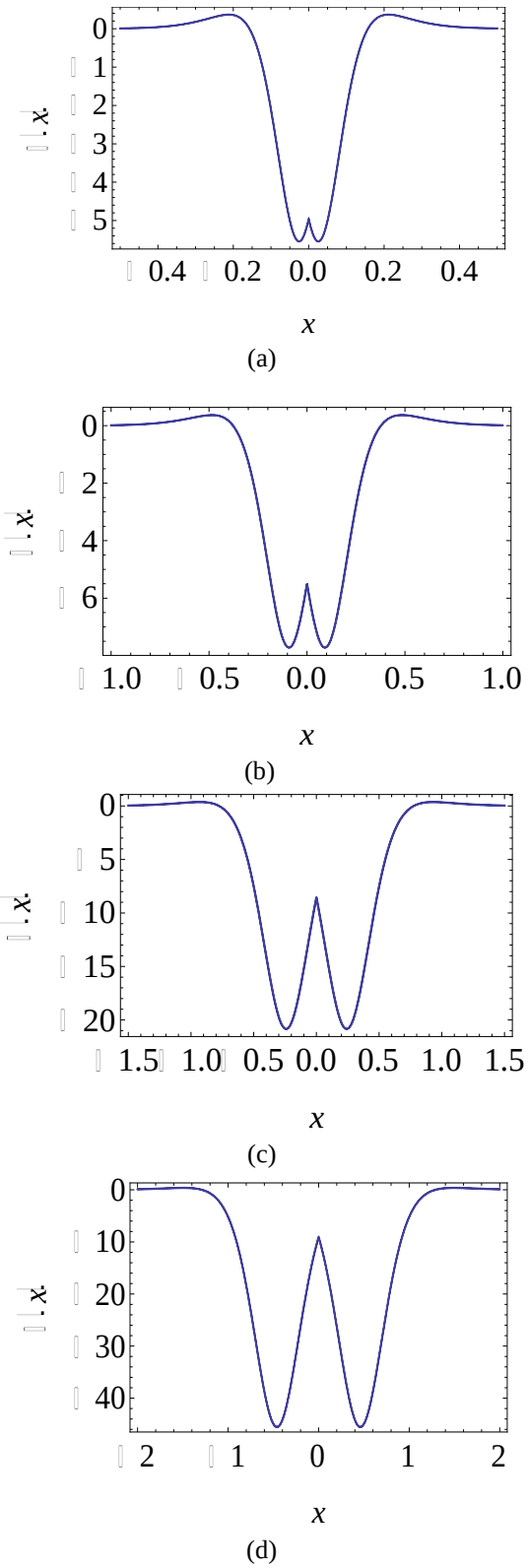

Fig 3: Information density in position space for isospectral Hydrogen potential with and deformation parameter (a) **α**=0.1, c = 0.2, (b) **α** = 0.2, c =0.3, (c) **α** = 0.3,c = 0.4, (d) **α** = 0.4, = 0.45

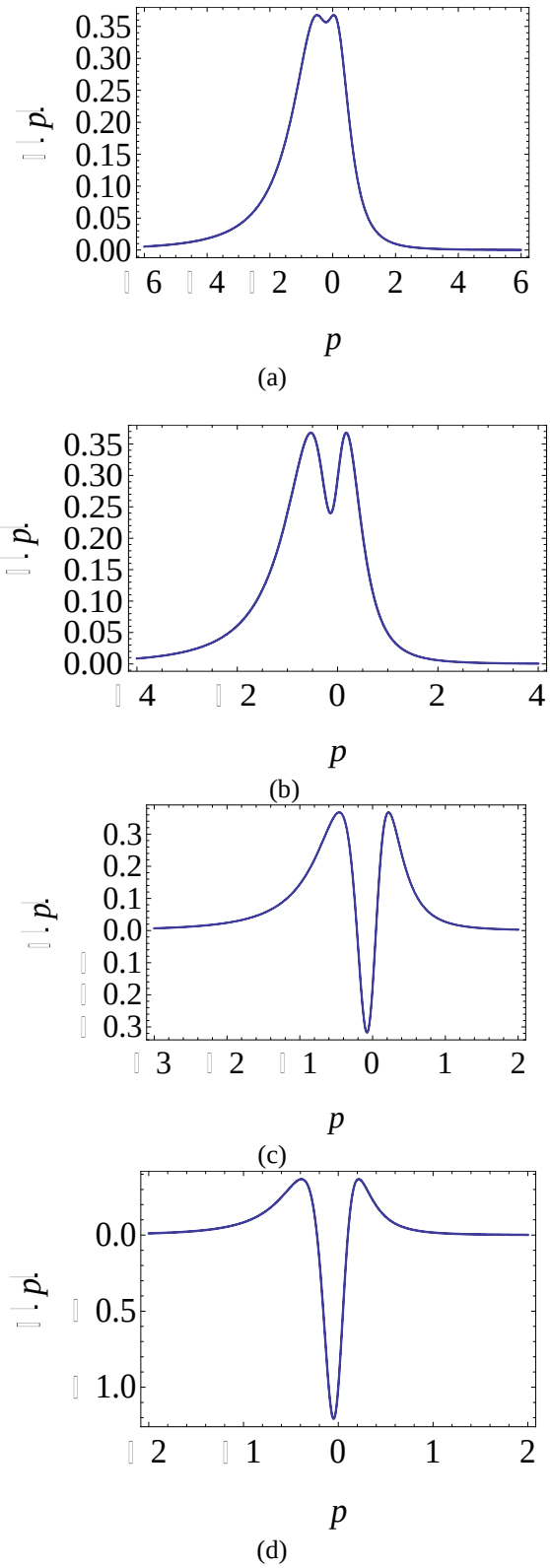

Fig 4: Information density in momentum space for isospectral Hydrogen potential with **α** and deformation parameter (a) **α**=1.5, c = 0.15, (b) **α**=2, c =0.2, (c) **α** =3, c=0.3, (d) **α** =4, c=0.4

The momentum probability density function of the hydrogen potential exhibits sharp distant peaks. The dip in the momentum density function changes with the parameter **α**. As seen in figures 1 and 3, the position space entropy densities have quite asymmetric shape which depends on the values of different parameters. It is interesting to observe the development of sharp dip on the peaks of density functions which increases with the deformation parameter characterizing specific properties of the potential. The momentum density function shown in figures 2 and 4 as function of different parameters also possess different behavior for various values of the parameter.

IV. CONCLUSION

The information density of quantum mechanical systems is a great scientific challenge as it provides a deeper insight into the internal structure of the systems. The isospectral wave functions are obtained in position space using isospectral Hamiltonian approach. The analytical form of corresponding wave functions in momentum space is obtained using Fourier transform of position space wave function. We have investigated the position and momentum space information density of isospectral hydrogen potential as a function of deformation parameter. The Some interesting features of information density are graphically demonstrated and their properties are analyzed.

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