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QUATERNION ANALYTICITY OF HARMONIC OSCILLATOR

Seema Rawat
Department of Physics
Zakir Hussain College, Jawahar Nehru Marg, New Delhi, India.
Email: rawatseema1@rediffmail.com

ABSTRACT

Theory of quaternionic harmonic oscillator has been analyzed for the system of bosons and fermions respectively in terms of commutation and anticommutation relations. Corresponding Eigenvalues of particle Hamiltonian and number operators are calculated by imposing the restriction on the component of quaternion variables. It has been shown that the supercharge commutes with bosonic and fermionic number operators while the Hamiltonian be constructed from anticommutation of supercharges. In SUSY algebra it has been shown that the supercharges commute with the supersymmetric Hamiltonian.

1. Introduction:

Quaternions were the very first example of hyper complex numbers having the significant impacts on mathematics and physics [1-3]. In quaternionic framework, natural laws [4] and basic equations have been formulated by Finkelstein et al [5-7]. Finkelisteinet.al. [1] first introduced quaternionic quantum oscillator. Duc. [8] used quaternions to construct bosonic oscillator. Since quaternions are the extension of complex numbers but they differs in the way that they do not commute among themselves. This puts restrictions in the formulations of bosonic quaternion oscillator, fermion oscillator and then supersymmetric harmonic oscillator. After the invention of spinning particles attempts were made to introduce quaternions in terms of Pauli matrices and accordingly the isomorphism between matrices and quaternions was developed by Jolly [9] &Silveria [10].

Supersymmetry was first appeared in field theories in terms of bosonic and fermionic fields and the possibility was early observed that it could accommodate a Grand Unified Theory (GUT) for four basic interactions of nature (strong, weak, electromagnetic and gravitational). Gelfand and Liktman [11] did the first work on super algebra in space-time within the framework of Poincare algebra. The SUSY algebra in quantum mechanics was initiated within the work of Nicolai [12]. The bosonic degrees of freedom are characterized by bosonic creation and annihilation operators obeying the commutation relations. Similarly fermionic degrees of freedom are described by fermioniccreation and annihilation operators and obey anticommutation relations. Nicolai's SUSY algebra was described as N = 1 SUSY algebra, has been extended by Witten [13]. The exact supersymmetry describes symmetry between bosonic and fermionic degree of freedom and is essential ingredient in grand unified theory. The structure of Lie algebra that incorporates commutation and anticommutation relations, characterizes a new type of symmetry, dynamical symmetry which is supersymmetry i.e. symmetry that converts bosonic part into fermionic part and fermionic part into bosonic part. Hamiltonian is one of the generator of this super algebra, remains invariant under such transformations. So that tremendous physical contents are included in it as it connects different quantum systems. In quantum mechanical system SUSY has been found to be very useful [14]. Exact SUSY implies exact fermion boson masses, which has not been observed so far.

Keeping in view the above points and starting from the most general consideration that bosonic operator are quaternion valued and using the fundamental commutation relation that bosonic operators satisfy, we have obtained the bosonic harmonic oscillator in terms of Hamiltonian with some restrictions in terms of quaternion coefficients. Quaternion annihilation and creation operator has been obtained by replacing the imaginary number i by quaternion unit e_j. It has been shown that the number operator commutes with Hamiltonian and hence is a constant of motion. The commutation relation between number operator and creation operator as well as annihilation operator are obtained consistently. Accordingly using the eigenvalues of creation, annihilation and number operators, the complete eigenvalue spectrum of bosonic harmonic oscillator has been obtained. It has been shown that the ground state energy of harmonic oscillator comes out to be $\frac{1}{2}\omega_B$, while that of first excited state becomes $\frac{3}{2}\omega_B$ and so on and this gives the usual results of algebra used for the system of bosons. It has been shown that those particles, which do not follow the Bose-Einstein statistics exactly and show deformation from this statistics, are described in terms of q-deformed algebra. Hence we have obtained the quaternionic q-deformed oscillators and it has been shown that in the limiting case q→1, q-deformation results overlap with the actual results of system of bosons.

Similarly we have described the quaternionicfermionic operator, which satisfy anticommutation relation. The fermionic harmonic oscillator has been obtained in terms of Hamiltonian and eigenvalue spectrum. It has been shown that the eigenvalue equation of fermionic number operator does not change the state and the fermion harmonic oscillator consists only two energy values $-\frac{1}{2}\omega_F \,\&+\frac{1}{2}\omega_F$. So it is possible to create one fermionic particle in one state and thus obey Pauli exclusion principle. Like bosons we have constructed q- deformed fermion harmonic oscillator for these particles which exactly do not follow the F.D. statistics and hence anticommutation relations in terms of deformation parameter are obtained consistently and corresponding eigenvalue spectrum is described.

Supercharges, the generators of supersymmetric transformations, are constructed from bosonic and fermionic creation and annihilation operators. It has been shown that the supercharge commutes with bosonic and fermionic number operators while the Hamiltonian be constructed from anticommutation of supercharges. In SUSY algebra it has been shown that the supercharges commute with the supersymmetric Hamiltonian. Accordingly we have constructed energy eigenvalue of Hamiltonian by summing the bosonic and fermionic energy eigenvalues. SUSY Hamiltonian has been represented in diagonal matrix form and then splitted into two super partners Hamiltonian H_+& H_-, to obtain the energy eigenvalue spectrum of quaternion SUSY oscillator.

2. *Quaternionic Bosonic Harmonic Oscillator:*

Let us define bosonic quaternion oscillator as the extension of complex oscillator having the decomposition

$$\hat{a} = \frac{1}{\sqrt{6}}\left[\hat{a}_0 + e_1\hat{a}_1 + e_2\hat{a}_2 + e_3\hat{a}_3\right] \quad (1)$$

where $\hat{a}_0, \hat{a}_1, \hat{a}_2$ & $\hat{a}_3$ are operators and e_1, e_2 & e_3 are quaternion units .Its conjugate is defined as

$$\hat{a}^\dagger = \frac{1}{\sqrt{6}}\left[\hat{a}_0 - e_1\hat{a}_1 - e_2\hat{a}_2 - e_3\hat{a}_3\right] \quad (2)$$

Like other oscillator let us start with the following fundamental boson commutation relation i.e.

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1, \quad \left[\hat{a}, \hat{a}\right] = 0 \quad \& \quad \left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0 \quad (3)$$

The condition $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ leads to

$$\frac{1}{6}\left[\hat{a}_0 + e_1\hat{a}_1 + e_2\hat{a}_2 + e_3\hat{a}_3, \hat{a}_0 - e_1\hat{a}_1 - e_2\hat{a}_2 - e_3\hat{a}_3\right] = 1$$

which gives

$$e_1\left[\hat{a}_1, \hat{a}_0\right] + e_2\left[\hat{a}_2, \hat{a}_0\right] + e_3\left[\hat{a}_3, \hat{a}_0\right] = 3 \quad (4)$$

This is possible only when, If we impose following conditions

$$\left[\hat{a}_1, \hat{a}_0\right] = e_1, \quad \left[\hat{a}_2, \hat{a}_0\right] = e_2 \quad \& \quad \left[\hat{a}_3, \hat{a}_0\right] = e_3 \quad (5)$$

where we assumed that e_1, e_2 & e_3 commute with bosonic components $\hat{a}_0, \hat{a}_1, \hat{a}_2$ & $\hat{a}_3$ Or in general

$$\left[\hat{a}_0, \hat{a}_A\right] = e_A \qquad (A = 1,2,3)$$

$$\left[\hat{a}_\mu, e_\nu\right] = 0 \qquad (\mu, \nu = 0,1,2,3). \quad (6)$$

Let us describe the Hamiltonian for bosonic harmonic oscillator

$$\hat{H}_B = \frac{\hat{P}^2}{2} + \frac{1}{2}\omega^2\hat{q}^2 \quad (7)$$

which can be written in terms of $\hat{a}$ & $\hat{a}^\dagger$ as

$$\hat{H}_B = \frac{1}{2}\omega_B(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = \omega_B(1/2 + \hat{a}^\dagger\hat{a}) = \omega_B\left(\hat{N}_B + \frac{1}{2}\right) \quad (8)$$

where $\hat{N}_B = \hat{a}^\dagger\hat{a}$ is called bosonic number operator. In terms of bosonic components $\hat{H}_B$ can be given as

$$\hat{H}_B = \frac{\omega_B}{6}\left(\hat{a}_0{}^2 + \hat{a}_1{}^2 + \hat{a}_2{}^2 + \hat{a}_3{}^2\right) \quad (9)$$

We can form the Hamiltonian by

$$\hat{a} = \frac{1}{\sqrt{2\omega}}(\omega\hat{q} - e_1\hat{p}) \,\&\, \hat{a}^\dagger = \frac{1}{\sqrt{2\omega}}(\omega\hat{q} + e_1\hat{p}) \quad (10)$$

Then to recover the ordinary commutation relation

$$\left[\hat{q}, \hat{P}\right] = e_1\hbar \quad (11)$$

We then get the value of $\hat{q}$ & $\hat{P}$ coordinate by

$$\hat{P} = \frac{1}{e_1}\sqrt{\frac{\omega_b}{2}}\left(-\hat{a}^\dagger + \hat{a}\right) = \sqrt{\frac{\omega_b}{3}}(-\hat{a}_1 + e_3\hat{a}_2 - e_2\hat{a}_3)$$
$$\hat{q} = \sqrt{\frac{2}{\omega_b}}\left(\hat{a}^\dagger + \hat{a}\right) = \sqrt{\frac{3}{\omega_b}}\hat{a}_0 \quad (12)$$

In the ordinary case,substituting $e_1 = i$, the commutation relation between $\hat{q}$ & $\hat{P}$ reduces to $\left[\hat{q}, \hat{P}\right] = i\hbar$.

The bosonic number operator $\hat{N}_B$ commutes with Hamiltonian $\hat{H}_B$ i.e.

$$[\hat{N}_B, \hat{H}_B] = 0 \tag{13}$$

which satisfies the following relations

$$[\hat{N}_B, \hat{a}^\dagger] = \left[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger\right] = \hat{a}^\dagger a \hat{a}^\dagger - \hat{a}^\dagger \hat{a}^\dagger \hat{a} = \hat{a}^\dagger$$
$$[\hat{N}_B, \hat{a}] = \left[\hat{a}^\dagger \hat{a}, \hat{a}\right] = \hat{a}^\dagger \hat{a}\hat{a} - \hat{a}\hat{a}^\dagger \hat{a} = -\hat{a}$$

$$[\hat{N}_B, \hat{N}_B] = 0$$

$$[\hat{N}_B, \hat{H}_B] = \omega_B \left(\hat{a}^\dagger \hat{a}, \hat{a}^\dagger \hat{a} + \frac{1}{2}\right) = 0$$

$$[\hat{H}_B, \hat{a}] = \omega_B \left(\hat{N}_B + \frac{1}{2}, \hat{a}\right) = -\omega_B \hat{a}$$

$$[\hat{H}_B, \hat{a}^\dagger] = \omega_B \hat{a}^\dagger \tag{14}$$

Hilbert space of the oscillator is defined by $\hat{a}^\dagger$ and $\hat{a}$ may be regarded as creation operator and annihilation operator respectively.Let

$$\hat{N}_B |n\rangle = n_B |n\rangle \quad with \quad \langle n|n\rangle \neq 0 \tag{15}$$

Then from equations (14)

$$\hat{N}_B a|n\rangle = a\left(\hat{N}_B - 1\right)|n\rangle = (n-1)a|n\rangle \tag{16}$$

Hence $a|n\rangle$ is an eigenvector of $\hat{\mathrm{N}}_\mathrm{B}$ with Eigen value $(n-1)$ provided only that $a|n\rangle \neq 0$. The squared norm of this vector is

$$\left(\langle n|\hat{a}^\dagger\right)\left(a|n\rangle\right) = \langle n|\hat{N}_B|n\rangle = n\langle n|n\rangle \tag{17}$$

From equation (14a) we get

$$\hat{N}_B \hat{a}^\dagger |n\rangle = \hat{a}^\dagger \left(\hat{N}_B + 1\right)|n\rangle = (n+1)\hat{a}^\dagger |n\rangle \tag{18}$$

The squared norm of this vector is

$$\left(\langle n|\hat{a}\right)\left(\hat{a}^\dagger |n\rangle\right) = \langle n|\hat{N}_B + 1|n\rangle = (n+1)\langle n|n\rangle \tag{19}$$

Which never vanishes because $n \geq 0$. Thus $\hat{a}^\dagger |n\rangle$ is an eigenvector of $\hat{N}_B$ with eigenvalue $(n+1)$. By repeatedly applying operator $\hat{a}^\dagger$ we can construct set of eigenvectors having the eigenvalues $n, n+1, n+2, n+3......$ The sequence will begin with eigenvalue $n = 0$.

So the orthonormal eigenvectors of $\hat{N}_B$ will be denoted as $|n\rangle$

$$\hat{N}_B |n\rangle = n_B |n\rangle, \qquad n_B = 0,1,2,3 - - - - - \tag{20a}$$

Since $\hat{a}^\dagger |n\rangle$ is proportional to $|n+1\rangle$. So we may write

$$\hat{a}^\dagger |n\rangle = C_n |n+1\rangle \tag{20b}$$

where C_n is the constant of proportionality, can be obtained from the norm of this vector. Hence

$$|C_n|^2 = \left(\langle n|\hat{a}\right)\left(\hat{a}^\dagger |n\rangle\right) = n+1 \Rightarrow C_n = \sqrt{n+1}$$
$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle \tag{21}$$

From this result we have

$$|n\rangle = (n!)^{-\frac{1}{2}} \left(\hat{a}^\dagger\right)^n |n+1\rangle \tag{22}$$

which describes the following expression for the matrix element associated with the creation and annihilation operators on imposing the orthonormality condition to equation (22)

$$\langle n'|\hat{a}^\dagger|n\rangle = \sqrt{n+1}\delta_{n',n+1} \tag{23a}$$

$$\langle n'|\hat{a}|n\rangle = \sqrt{n}\delta_{n',n-1} \tag{23b}$$

Shows that matrix has its non-zero element one space below the diagonal and one space above the diagonal. From equations (22a) and (23b), we get following results

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ & $\hat{a}|0\rangle$, $n \neq 0$ (24a)

Finally the eigenvalues and eigenvectors of bosonic harmonic oscillator are described as

$$\hat{H}_B|n\rangle = E_B|n\rangle \quad (24b)$$

where $E_B = \left(n + \frac{1}{2}\right)\omega_B$. In other words we get following eigen-value spectrum

$$\hat{a}|0\rangle = 0,\ \hat{a}^\dagger|n\rangle = \sqrt{(n+1)}|n+1\rangle, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle\ \hat{N}_B|n\rangle = n|n\rangle \quad (25)$$

The state vectors as then span the Hilbert space

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle \quad (26)$$

Where $|0\rangle$ is considered as ground or vacuum state, which must be normalized

$$\langle 0|0\rangle = 1 \quad (27)$$

Then gives the familiar results

$$\hat{H}_B|n\rangle = E_B|n\rangle = \omega_B(n_B + 1/2)\ |n\rangle \quad E_B = \omega_B\left(n_B + \frac{1}{2}\right) \quad (28)$$

where $n_B = 0,1,2,3,4$...........

Now one can construct the eigenvalue spectrum as

$$\hat{H}_B|0\rangle = \omega_B\left(\hat{a}^\dagger\hat{a}|0\rangle + 1/2|0\rangle\right)$$

$$E_{B,0} = \frac{1}{2}\omega_B \quad (29a)$$

This is the ground state energy of bosonic harmonic oscillator

Now the energy of first excited state is given by

$$\hat{H}_B|1\rangle = \omega_B\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|1\rangle = \omega_B\left(1 + \frac{1}{2}\right)|1\rangle = \frac{3}{2}\omega_B|1\rangle$$

$$E_{B,1} = \frac{3}{2}\omega_B \quad (29b)$$

Similarly.

$$E_{B,2} = \frac{5}{2}\omega_B \quad (29c)$$

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and so on. It shows that Eigen value is $+\frac{1}{2}$ for ground state.

We may now express the Hamiltonian $\hat{H}_B$ in the representation in which $\hat{q}$ is diagonal, we can calculate the eigenvalues of $\hat{H}_B$. The eigenfunction $\psi_n(x) = \langle n|x\rangle$ can be taken as expansion coefficient of abstract eigenvectors of $\hat{H}_B$ and $|n\rangle$ in terms of eigenvectors of $\hat{q}$ .We may now calculate the matrix element spanned by the basis of elements formed by the eigenvectors of $\hat{H}_B$, and of $\hat{N}_B = (\hat{a}^\dagger\hat{a})$ in the following matrix form using equation (12) for projection operator and equation (23) i.e.

$$\langle n'|q|n\rangle = \sqrt{\frac{2}{\omega_B}}\begin{bmatrix} 0 & \sqrt{1} & o & 0 \text{ ....} \\ \sqrt{1} & 0 & \sqrt{2} & 0 \text{ ....} \\ 0 & \sqrt{2} & 0 & `\sqrt{3} \text{ ....} \\ 0 & 0 & \sqrt{3} & 0 \text{ ....} \end{bmatrix} \quad (30)$$

3. *q-Deformed Bosonic Oscillator:*

Sometimes particles do not follow Bose Einstein's statistics completely. They show deformation from it. In that case they are defined by introducing the deformation parameter. Suppose it is denoted by q . Then the Bosons are supposed to follow the following relations

$$[\hat{a}, \hat{a}^\dagger]_q = \hat{a}\hat{a}^\dagger - q\hat{a}^\dagger\hat{a} = q^{c\hat{N}_B} \quad (31)$$

where $\hat{N}_B$ is the bosonic number operator. In symplectic representation $\hat{a}, \hat{a}^\dagger$ can be defined as

$$\hat{a} = \hat{a}_\alpha + e_2\hat{a}_\beta \ , \qquad \hat{a}^\dagger = \hat{a}_\alpha + e_2\hat{a}_\beta$$

$$\hat{a}^\dagger\hat{a} = \hat{a}_\alpha{}^2 + \hat{a}_\beta{}^2 + e_2\left[\hat{a}_\alpha, \hat{a}_\beta\right]$$
$$\hat{a}\hat{a}^\dagger = \hat{a}_\alpha{}^2 + \hat{a}_\beta{}^2 - e_2\left[\hat{a}_\alpha, \hat{a}_\beta\right] \qquad (32)$$

So equation (31) can be written as

$$\left(\hat{a}^\dagger{}_\alpha{}^2 + \hat{a}^\dagger{}_\beta{}^2\right)(1-q) - e_2\left[\hat{a}^\dagger{}_\alpha, \hat{a}^\dagger{}_\beta\right](1+q) = q^{c\hat{N}_B} \ (33)$$

The notation in terms of deformation parameter q can be defined as

$$[x]_q{}^c = \frac{q^x - q^{cx}}{q - q^c} \qquad (34)$$

$$\hat{a}^\dagger\hat{a} = \left[\hat{N}_B\right]_q{}^c = \frac{q^{\hat{N}_B} - q^{c\hat{N}_B}}{q - q^c} \qquad (35a)$$

$$\hat{a}\hat{a}^\dagger = \left[\hat{N}_B + I\right]_q{}^c = \frac{q^{\hat{N}_B + I} - q^{c\left(\hat{N}_B + I\right)}}{q - q^c} \qquad (35b)$$

Then the relation (9b) and (10a) in terms of deformation parameter can be written as.

$$\hat{H}_B = \frac{\omega_B}{2}\left(\left[\hat{N}_B + I\right]_q{}^c + \left[\hat{N}_B\right]_q{}^c\right) \qquad (36)$$

$$\left[\hat{P}, \hat{q}\right] = -e_1\left(\left[\hat{N}_B + I\right]_q{}^c + \left[\hat{N}_B\right]_q{}^c\right) \qquad (37)$$

and the number operator $\hat{N}_B$ is given by equation

$$\hat{N}_B|n\rangle = n|n\rangle$$

Hence equation (24b) and (25) can be written as

$$\hat{E}_B = \frac{\omega_B}{2}\left([n+1]_q{}^c + [n]_q{}^c\right) \qquad (38)$$

$$\left[\hat{P}, \hat{q}\right] = -e_1\left([n+1]_q{}^c + [n]_q{}^c\right) \quad (39)$$

Taking $c = -1$ the energy of q-deformed oscillator is given as follows

$$\hat{E} = \frac{\omega_b}{2}\left[\frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} + \frac{q^n - q^{-n}}{q - q^{-1}}\right] (40)$$

As such we may have the following set of equation of energy spectrum i.e. for ground state energy

For $n = 0$ $E_{B,0} = \frac{1}{2}\omega_B$ (41)

Corresponding to the usual results of quantum mechanics.

(i) For $n = 1$

$$E_{B,1} = \frac{1}{2}\omega_B\left[\frac{q^2 + q - q^{-2} - q^{-1}}{q - q^{-1}}\right] = \frac{1}{2}\omega_B\left[\frac{\left(q - q^{-1}\right) + \left(q - q^{-1}\right)\left(q + q^{-1}\right)}{q - q^{-1}}\right]$$

$$E_{B,1} = \frac{1}{2}\omega_B\left(1 + q + q^{-1}\right) \qquad (42)$$

(ii) For $n = 2$

$$E_{B,2} = \frac{1}{2}\omega_B\left[\frac{q^3 + q^2 - q^{-3} - q^{-2}}{q - q^{-1}}\right]$$
$$= \frac{1}{2}\omega_B\left[\frac{\left(q - q^{-1}\right)\left(q^2 + q^{-2} + 1\right) + \left(q - q^{-1}\right)\left(q + q^{-1}\right)}{q - q^{-1}}\right]$$

$$E_{B,2} = \frac{1}{2}\omega_B\left(1 + q + q^{-1} + q^2 + q^{-2}\right) \qquad 43)$$

and so on.

In the limiting case $q \to 1$, we get the familiar results, but when $q \neq 1$ the energy states are not equally spaced, although we get well known results that energy is 1/2 for ground state still.

4. *Quaternionic Fermionic Harmonic Oscillator:*

Similarly we can write the following anti commutation relation for fermionic harmonic oscillator

$$\{\hat{b},\hat{b}^{\dagger}\}=1 \qquad \{\hat{b},\hat{b}\}=0 \qquad \{\hat{b}^{\dagger},\hat{b}^{\dagger}\}=0 \quad (44)$$

where $\hat{b}$ is a quaternion and may be decomposed as

$$\hat{b}=\frac{1}{\sqrt{6}}(\hat{b}_0+e_1\hat{b}_1+e_2\hat{b}_2+e_3\hat{b}_3)=\hat{b}_\alpha+e_2\hat{b}_\beta$$
$$\hat{b}^{\dagger}=\frac{1}{\sqrt{6}}(\hat{b}_0-e_1\hat{b}_1-e_2\hat{b}_2-e_3\hat{b}_3)=\hat{b}_\alpha-e_2\hat{b}_\beta \quad (45)$$

where $\hat{b}_\alpha=\hat{b}_0+e_1\hat{b}_1$ and $\hat{b}_\beta=\hat{b}_2-e_2\hat{b}_3$.

Now for $\hat{b}$ $\hat{b}^{\dagger}$ to be the component of fermionic harmonic oscillator the necessary and sufficient condition that these must have to satisfy all the relations of equation (44). So we may apply these relations in following steps.

(i) The condition $\{\hat{b},\hat{b}^{\dagger}\}=1\Rightarrow$

$$\{\hat{b},\hat{b}^{\dagger}\}=\frac{1}{6}\{\hat{b}_0+e_1\hat{b}_1+e_2\hat{b}_2+e_3\hat{b}_3,\hat{b}_0-e_1\hat{b}_1-e_2\hat{b}_2-e_3\hat{b}_3\}$$
$$=\frac{1}{6}\left[2\hat{b}_0^{\,2}+2\hat{b}_1^{\,2}+2b_2^{\,2}+2b_3^{\,2}\right]$$

or $\hat{b}_0^{\,2}+\hat{b}_1^{\,2}+\hat{b}_2^{\,2}+\hat{b}_3^{\,2}=3$ (46)

(ii)The condition

$\{\hat{b},\hat{b}\}=0 \quad \Rightarrow \quad \hat{b}^2=0$ i.e.

$$\frac{1}{\sqrt{6}}(\hat{b}_0+e_1\hat{b}_1+e_2\hat{b}_2+e_3\hat{b}_3)\times\frac{1}{\sqrt{6}}(\hat{b}_0+e_1\hat{b}_1+e_2\hat{b}_2+e_3\hat{b}_3)=0$$

$$\Rightarrow \hat{b}_0^{\,2}-\hat{b}_1^{\,2}-\hat{b}_2^{\,2}-\hat{b}_3^{\,2}+e_1\{\hat{b}_0,\hat{b}_1\}+e_2\{\hat{b}_0,\hat{b}_2\}$$
$$+e_3\{\hat{b}_0,\hat{b}_3\}+e_1[\hat{b}_2,\hat{b}_3]+e_2[\hat{b}_3,\hat{b}_1]+e_3[\hat{b}_1,\hat{b}_3]=0 \quad (47)$$

(iii) The condition

$\{\hat{b}^{\dagger},\hat{b}^{\dagger}\}=0 \quad \Rightarrow \quad \hat{b}^{\dagger 2}=0$ i.e.

$$\frac{1}{\sqrt{6}}(\hat{b}_0-e_1\hat{b}_1-e_2\hat{b}_2-e_3\hat{b}_3)\times\frac{1}{\sqrt{6}}(\hat{b}_0-e_1\hat{b}_1-e_2\hat{b}_2-e_3\hat{b}_3)=0$$

$$\Rightarrow \hat{b}_0^{\,2}-\hat{b}_1^{\,2}-\hat{b}_2^{\,2}-\hat{b}_3^{\,2}-e_1\{\hat{b}_0,\hat{b}_1\}-e_2\{\hat{b}_0,\hat{b}_2\}$$
$$-e_3\{\hat{b}_0,\hat{b}_3\}+e_1[\hat{b}_2,\hat{b}_3]+e_2[\hat{b}_3,\hat{b}_1]+e_3[\hat{b}_1,\hat{b}_2]=0 \quad (48)$$

By combining equation (46), (47), (48) we get following conditions

$$\hat{b}_0^{\,2}+\hat{b}_1^{\,2}+\hat{b}_2^{\,2}+\hat{b}_3^{\,2}=3 \quad \hat{b}_0^{\,2}-\hat{b}_1^{\,2}-\hat{b}_2^{\,2}-\hat{b}_3^{\,2}=0$$
$$\{\hat{b}_0,\hat{b}_1\}=\{\hat{b}_0,\hat{b}_2\}=\{\hat{b}_0,\hat{b}_3\}=0 \quad \Rightarrow \{\hat{b}_0,\hat{b}_i\}=0\ ,\ i=1,2,3$$
$$[\hat{b}_2,\hat{b}_3]=[\hat{b}_3,\hat{b}_1]=[\hat{b}_1,\hat{b}_2]=0 \quad \Rightarrow [\hat{b}_i,\hat{b}_j]=0 \quad i\neq j \quad (49)$$

where the capital bracket defines the commutator and the curly bracket defines the anticommutator. Let us now write the fermionic Hamiltonian as

$$\hat{H}_F=\frac{1}{2}\omega_F(\hat{b}^{\dagger}\hat{b}-\hat{b}\hat{b}^{\dagger})=\frac{1}{2}\omega_F[\hat{b}^{\dagger},\hat{b}]$$
$$=\omega_F\left(\hat{b}^{\dagger}\hat{b}-\frac{1}{2}\right)=\omega_F\left(\hat{N}_F-\frac{1}{2}\right) \quad (50)$$

Substituting the value of $\hat{b}^{\dagger}\hat{b}$ in equation (50) from equation (45).

$$\hat{H}_F=\frac{2\omega_F\hat{b}_0}{6}(e_1\hat{b}_1+e_2\hat{b}_2+e_3\hat{b}_3) \quad (51)$$

We may express the fermions in terms of Pauli matrices as

$$\{\hat{b},\hat{b}^{\dagger}\}=I \quad \& \quad [\hat{b},\hat{b}^{\dagger}]=\sigma_3 \quad (52a)$$

By expanding the commutating and anticommutating bracket we get

$$\hat{b}\hat{b}^{\dagger}+\hat{b}^{\dagger}\hat{b}=\begin{bmatrix}1&0\\0&1\end{bmatrix} \& \hat{b}\hat{b}^{\dagger}-\hat{b}^{\dagger}\hat{b}=\begin{bmatrix}1&0\\0&-1\end{bmatrix} (52b)$$

$$\hat{b}\hat{b}^{\dagger}=\begin{bmatrix}1&0\\0&0\end{bmatrix} \& \hat{b}^{\dagger}\hat{b}=\begin{bmatrix}0&0\\0&1\end{bmatrix} \quad (52c)$$

Here $\hat{b}$ and $\hat{b}^{\dagger}$ can be formed from quaternion matrices as follows

$$\hat{b} = \frac{1}{2}(ie_1 - e_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (53a)$$

$$\hat{b}^\dagger = \frac{1}{2}(ie_1 + e_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad (53b)$$

Let us obtain the eigen value spectrum of fermion oscillator for which the Hilbert space with eigenvector $|n\rangle$ can be constructed as

$$\hat{N}_F|n\rangle = n|n\rangle\,;\ \hat{b}|1\rangle = |0\rangle \quad ;\hat{b}^\dagger|0\rangle = |1\rangle \quad \& \quad \hat{b}^\dagger|1\rangle = |0\rangle = \hat{b}|0\rangle \quad (54)$$

So that the eigen value equation (50) is given as

$$\hat{H}_\mathrm{F}|1\rangle = \omega_F\left(\hat{b}^\dagger\hat{b} - \frac{1}{2}\right)|1\rangle = \omega_F\left(1 - \frac{1}{2}\right)|1\rangle = \omega_F\,\frac{1}{2}|1\rangle$$

or $$E_F = \frac{1}{2}\omega_F$$

(55a)

$$\hat{H}_\mathrm{F}|0\rangle = \omega_F\left(\hat{b}^\dagger\hat{b} - \frac{1}{2}\right)|0\rangle = \omega_F\left(0 - \frac{1}{2}\right)|1\rangle = -\omega_F\,\frac{1}{2}|1\rangle$$

or $$\hat{E}_F = -\frac{1}{2}\omega_F$$

(55b)

also $\langle 0|0\rangle = 1$ shows the normalized state.Hence we get the eigenstate $-\frac{1}{2}\,\&+\frac{1}{2}$ in this oscillator.The ground state energy of this oscillator is $-\frac{1}{2}\omega_F$. Since $\{\hat{b}^\dagger,\hat{b}^\dagger\}=0$. So it is possible to create one fermionic particle in one state. Thus obey Pauli exclusion principle.

5. ***q-Deformed Fermionic Oscillator:***

We can now establish the deformaton for the system of fermions on the same basis as we have discussed earlier for the system of bosons. The number operator $\hat{N}_F$ takes values 0,1 and fermions satisfy the relation

$$\hat{b}\hat{b}^\dagger + q^{-1}\hat{b}^\dagger\hat{b} = q^{C\hat{N}_F} \quad (56a)$$

instead of anticomutation relation given by equation (44). Thus we get the following relation for fermion operators

$$\hat{b}^\dagger\hat{b} = \left[\hat{N}_F\right] \quad (56b)$$

and the fermionic Hamiltonian is described by equation (50). The spectra of which have only two levels

$$E_0 = -\frac{\omega_F}{2} \quad \& \, E_1 = \frac{\omega_F}{2} \quad (57)$$

The energies are now q-independent and ground state energy is thus negative $\left(-\frac{1}{2}\right)$.

6. ***Supersymmetric Harmonic Oscillator:***

Let us now construct a simple supersymmetric quantum mechanical system that include the bosonic oscillator degree of freedom $(\hat{a}^\dagger, \hat{a})$ and fermionic spin $\left(-\frac{1}{2}\right)$ degrees of freedom $(\hat{b}^\dagger, \hat{b})$ .We call it as supersymmetric harmonic oscillator .The supersymmetry is obtained by annihilating simultaneously one bosonic quantum $n_b \to n_b - 1$ and creating one fermionic quantum $n_f \to n_f - 1$ or vice versa. We illustrate the annihilating (supersymmetric) charges (generators) for SUSY oscillator by letting $\omega_B = \omega_F = \omega$ as

$$\hat{Q} = \sqrt{\omega}\left(\hat{a}^\dagger \hat{b}^\dagger\right) \qquad \hat{Q}^\dagger = \sqrt{\omega}\left(\hat{b}^\dagger \hat{a}\right) \quad (58)$$

where $\hat{Q}$ & $\hat{Q}^\dagger$ satisfies

$$\left[\hat{Q}, \hat{N}_B\right] = 0 \qquad \left[\hat{Q}^\dagger, \hat{N}_B\right] = \hat{Q}^\dagger$$
$$\left[\hat{Q}, \hat{N}_F\right] = 0 \qquad \left[\hat{Q}^\dagger, \hat{N}_F\right] = -\hat{Q}^\dagger \quad (59a)$$

and

$$\hat{Q}\left|n_B, n_F\right\rangle = \sqrt{(n_B+1)}\left|n_B+1, n_F-1\right\rangle \qquad \text{if } n_F = 1$$
$$= 0 \qquad \text{if } n_F = 0$$

along with

$$\hat{Q}_+\left|n_B, n_F\right\rangle = \frac{1}{\sqrt{n_B}}\left|n_B-1, n_F+1\right\rangle \text{ if } n_B \neq 0, n_F = 0$$
$$= 0 \quad \text{if } n_B = 0 n_F = 1 \quad (59b)$$

Thus energy states $\left|n_B+1, n_F-1\right\rangle$ and $\left|n_B-1, n_F+1\right\rangle$ are degenerate in energy with the state $\left|n_B, n_F\right\rangle$. So that Hamiltonian $\hat{H}$ becomes

$$\hat{H} = \left\{\hat{Q}^\dagger, \hat{Q}\right\} = \omega\left\{\hat{a}^\dagger \hat{b}\hat{b}^\dagger \hat{a} + \hat{b}^\dagger \hat{a}\hat{a}^\dagger \hat{b}\right\} = \omega\left\{\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}\right\}$$
$$= \hat{H}_{osc} + \hat{H}_{spin} = \hat{H}_{Boson} + \hat{H}_{Fermion} = \omega\left(\hat{N}_B + \hat{N}_F\right) \quad (59c)$$

and

$$\left[\hat{H}, \hat{Q}\right] = \left[\hat{H}, \hat{Q}^\dagger\right] = 0 \quad (59d)$$

Energy eigenvalue is

$$E = \omega\left(n_B + n_F\right) = \omega n \quad (60)$$

$n_F = 0,1$ & $n_B = 0,1,2,3$...............

In terms of quaternion component the total oscillator is written as

$$\hat{H} = \hat{H}_B + \hat{H}_F$$
$$= \frac{\omega}{6}\left(\hat{a}_0^{\,2} + \hat{a}_1^{\,2} + \hat{a}_2^{\,2} + \hat{a}_3^{\,2} + 2\hat{b}_0(\mathrm{e}_1\hat{b}_1 + \mathrm{e}_2\hat{b}_2 + \mathrm{e}_3\hat{b}_3)\right) \quad (61)$$

which can be visualized analogously to the following expression of harmonic oscillator in one dimensional form discussed earlier [44] i.e.

$$\hat{H} = \left(-\frac{d^2}{dx^2} + \frac{x^2}{4}\right) - \frac{1}{2}\left[\psi, \psi^\dagger\right] \quad (62)$$

where

$$\hat{H}_B = \left(-\frac{d^2}{dx^2} + \frac{x^2}{4}\right) \quad \& \quad \hat{H}_F = -\frac{1}{2}\left[\psi, \psi^\dagger\right].$$

Here the term $2\hat{b}_0(\mathrm{e}_1\hat{b}_1 + \mathrm{e}_2\hat{b}_2 + \mathrm{e}_3\hat{b}_3)$ given in equation (61) removes the zero point energy. But non-linear.in nature hence in the general case we can write susy Hamiltonian in the form

$$\hat{H} = \left(-\frac{d^2}{dx^2} + w^2\right) - \left[\psi, \psi^\dagger\right] w' \quad (63)$$

In order to write the explicit form of a general supersymmetric harmonic oscillator Hamiltonian in three dimensional representation and accordingly to visualize the present theory of quaternionic harmonic oscillator in three dimension ,we may substitute the following relations between the operators;

$$\hat{a}_0^{\,2} = -\frac{6}{\omega}W^2, \quad \hat{a}_1^{\,2} = -\frac{6}{\omega}\frac{d^2}{dx^2},$$
$$\hat{a}_2^{\,2} = -\frac{6}{\omega}\frac{d^2}{dy^2}, \quad \hat{a}_3^{\,2} = -\frac{6}{\omega}\frac{d^2}{dz^2},$$

$$\hat{b}_0\hat{b}_1 = -\sqrt{\frac{\omega}{3}}\sigma_3 W'(x), \quad \hat{b}_0\hat{b}_2 = -\sqrt{\frac{\omega}{3}}\sigma_3 W'(y),$$
$$\hat{b}_0\hat{b}_3 = -\sqrt{\frac{\omega}{3}}\sigma_3 W'(y) \quad (64)$$

As such the SUSY Harmonic oscillator becomes

$$\hat{H} = \left(-\frac{d^2}{dx^2} - \frac{d^2}{dy^2} - \frac{d^2}{dz^2} - W^2 \right) - \sigma_3 \{ e_1 W'(x) + e_2 W'(y) + e_3 W'(z) \} \quad (65)$$

This is three-dimensional representation of quaternionic harmonic oscillator. and can be reduced further to ther case of one dimension only by setting $Y = Z = 0$ i.e.one-dimensional harmonic oscillator is

$$\hat{H} = \left(-\frac{d^2}{dx^2} - w^2 \right) - \sigma_3 (e_2 W'(x))$$
$$= \left(-\frac{d^2}{dx^2} - w^2 \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} (e_2 W'(x)) \quad (66)$$

$$= \begin{bmatrix} \left(-\frac{d^2}{dx^2} - w^2 - e_2 W'(x) \right) & 0 \\ 0 & \left(-\frac{d^2}{dx^2} - w^2 + e_2 W'(x) \right) \end{bmatrix} \quad (67)$$
$$= \begin{bmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{bmatrix}$$

where $\hat{H}_+$ and $\hat{H}_-$ are denoted as quaternionicsuperpartner Hamiltonians i.e.

$$\hat{H} = \begin{bmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{bmatrix} \quad (68)$$

where A is generalized (combined bosonic and fermionic i.e. supersymmetric) annihilation operator and $A^\dagger$ is generalized creation operator, and given by

$$A = W(x) + e_2 \frac{d}{dx} \quad (69)$$

$$A^\dagger = -W(x) + e_2 \frac{d}{dx} \quad (70)$$

The supercharges given by equation (58) may thus be represented by (10) and (58) respectively for bosonic and fermionic operators along with the expression used by equation (12) for position and momentum operators and accordingly we may thus obtain the supersymmetric Hamiltonian operator given by equation (66). Returning to equation (60), the eigen state is described as $|n_B, n_F\rangle$ and ground state as $|0,0\rangle$ so that

$$\hat{\mathrm{H}} |n_B, n_F\rangle = \mathrm{E}_{n_B, n_F} |n_B, n_F\rangle \qquad (n_B = 0,1,2,3, \quad ; \; n_F = 0 \text{ or } 1) \quad (71)$$

We also have

$$\hat{Q} |n, 1\rangle = \sqrt{n+1} |n+1, 0\rangle$$
$$\hat{Q}^\dagger |n+1, 0\rangle = \sqrt{n+1} |n, 1\rangle \quad (72)$$

These supercharges represents conversion of a fermionic state to a bosonic state and bosonic state to fermionic state

$$\hat{Q}^\dagger |\text{boson}\rangle = |\text{fermion}\rangle$$
$$\hat{Q} |\text{fermion}\rangle = |\text{boson}\rangle \quad (73)$$

Equation (59c) is the direct analogous of following equations of super symmetry

$$\{ \hat{Q}_\alpha^\dagger, \hat{Q}_\beta \} = \mathrm{P}^\mu (\sigma_\mu)_{\alpha\beta} \quad (74)$$

$$\left[\hat{H}, \hat{Q}_\alpha \right] = 0 \quad (75)$$

For $\mu = 0$ and $\alpha = \beta = 1$. Supercharges always commute with usual Hamiltonian. Thus the anticommuting charges in quaternion formalism combine to form the generators of time translation, namely the Hamiltonian $\hat{H}$.The ground state of this system is the state $|0\rangle_{ośc} |0\rangle_{spin}$ or $|0\rangle_{boson} |0\rangle_{fermion} = |0,0\rangle$, where both bosonic and fermionic degrees of freedom are in the lowest energy state. This state is unique one and satiafies

$$\hat{Q}|0,0\rangle = \hat{Q}^{\dagger}|0,0\rangle = 0 \qquad (76)$$

As such we may calculate the energy of supersymmetric harmonic oscillator from equation (3.59) i.e.

$$\hat{H}|0,0\rangle = \omega\left(\hat{a}^{\dagger}\hat{a} + \hat{b}^{\dagger}\hat{b}\right)|0,0\rangle = 0 \qquad (77)$$

which shows that SUSY does not break for ground state, and we have the higher energy states in the following manner

$$\hat{H}|1,0\rangle = \omega\left(\hat{a}^{\dagger}\hat{a} + \hat{b}^{\dagger}\hat{b}\right)|1,0\rangle = \omega \qquad (78a)$$

$$\hat{H}|0,1\rangle = \omega\left(\hat{a}^{\dagger}\hat{a} + \hat{b}^{\dagger}\hat{b}\right)|0,1\rangle = \omega \qquad (78b)$$

and accordingly

$$\hat{H}|1,0\rangle = E|1,0\rangle = \omega \qquad (78c)$$

$$\hat{H}|2,0\rangle = E|1,1\rangle = 2\omega \qquad (78d)$$

$$\hat{H}|3,0\rangle = E|2,1\rangle = 3\omega \qquad (78e)$$

....

and so on. It shows that the energy states (1,0) and (0,1) are degenerate states. Similarly (2,0) and (1,1), (3,0), (2,1) and (1,2) are also degenerate. As such the excited states form a tower of degenerate levels (table) with energy $\left(n+\frac{1}{2}\right)\hbar\omega \pm \frac{1}{2}\hbar\omega$, where the sign of the second term is determined by whether the spin $\frac{1}{2}$ state is $|1\rangle$ (plus) or $|0\rangle$ (minus) .

Illustration as follows

Energy	**State**	
	Boson	**Fermion**
0	$\|0,0\rangle$	
ω	$\|1,0\rangle$	$\|0,1\rangle$
2ω	$\|2,0\rangle$	$\|1,1\rangle$
3ω	$\|3,0\rangle$	$\|2,1\rangle$
4ω	$\|4,0\rangle$	$\|3,1\rangle$

The tower of states describes the boson fermion degenracy for exact supersymmetry. The bosonic state $|n+1,0\rangle$ (called bosonic in field theory analogy because they contain no fermions) have the same energy as their fermion partner in $|n,1\rangle$. Thus the quaternion reformation of a super symmetry gives rise to a simple representation of super symmetry in quantum mechanics. It is however, trivial since it describes non-interacting boson (oscillator) and fermions (spin –1/2 particles).

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