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Stability Analysis of a Simultaneously Precessing, Spinning and Nutating Beam

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Abstract

In this paper, the stability of a simultaneously precessing, nutating and spinning beam with a tip mass is analyzed for four sets of initial conditions, which can be linearly combined to serve as the input for any form of system initial condition. Generalized equations of motion for the system are derived using Newton's second law of motion and these are transformed into an Eigen value problem and solved using principles of linear algebra. Centrifugal stiffening term is added into the equation following zero deformation along the longitudinal axis of the beam. The system response is obtained for the four initial conditions and the stability criterions are presented. Margins of stability are constructed for several system parameters. The system stability is governed by internal parameters, like Young's Modulus, density, geometry, etc. The current work specifically analyzes special cases like that of a simultaneously precessing and spinning beam, a simultaneously nutating and spinning beam, a simultaneously nutating and precessing beam, a nutating beam, a precessing beam and a spinning beam. This approach is undertaken to give insight into the system dynamics and to understand the effect of different parameters on the system stability. Conclusions are reached during the analysis of the response function. It is established that, a simultaneously nutating and spinning beam is always bounded and stable. A simultaneously precessing and nutating beam is stable except for certain range of nutation angle. A simultaneously precessing and spinning beam is always unbounded and unstable. A nutating beam is found to be always stable.

Keywords: Precession, Nutation, Spinning, Centrifugal Stiffening, Response Function

INTRODUCTION

Mechanical, aerospace and civil engineers often have to deal with structures undergoing rotation about a point, particularly in case of mast antenna structures, robotic arms for material handling, helicopter, etc. Hence, it is very important to simultaneously analyze precession, spin and nutation. This paper is primarily directed towards solving this problem. Earlier, Nandi A and Neogy S [1] developed a methodology for analyzing the stability of non-symmetric rotor in a rotating frame. Later, Nandi A et al. [2] extended the methodology to that of the stability analysis of a flexible spinning and precessing rotor with non-symmetric shaft. Nandi A et al. [3] deduced the generalized equations of rotation about a point and simulated the same. Bose S et al. [4] analyzed the stability criterion of a spinning and precessing Viscoelastic rotor model under the effect of tensile centrifugal force. Instead of only elasticity, viscous effect was also considered. The effect of tensile centrifugal force on system stability was also studied. Bose S et al. [5] studied similar rotor models without the effect of tensile centrifugal force. Bose S et al. [6] also developed different analytical and finite element models for analyzing a flexible spinning and precessing rotor. Then, Nandi A et al. [7] developed the governing equations of motion for a precessing and nutating beam with a tip mass. The stability criterion was developed using a variant of Hill's method. Later, Bose S et al. [8] studied the Finite Element analysis of distributed precession-softening of a spinning and precessing rotor. Very recently, Bose S et al. [9] studied the stability of a precessing and nutating Viscoelastic beam with a tip mass. Though significant research works have been reported on rotating beams and spinning rotors, the literatures on analysis of a simultaneously precessing, nutating and spinning beam with a tip mass are scarce. Hence, an initiative is undertaken to understand the dynamics of such a system. The methodology and the mathematical formulation have been developed first. Then, the response is obtained, which is studied for four sets of initial conditions. Finally, the stability margins are developed and the conclusions are derived at for specific cases mentioned earlier.

MATHEMATICAL MODELING AND FORMULATION

Figure 1 shows the beam with tip mass in motion. Though only precessing and nutating motion is shown, it should be remembered that the beam also has spinning motion. The y-axis is aligned with the centerline of the beam. The mass of the beam is assumed to be negligible in comparison to the tip mass. The position vector of the tip mass is represented by ρ in Equation (1). The expression of angular speed is given by Equation (2) and this is justified by Figure 1.

Table 1. List of Symbols and their meanings

Symbols	Meanings
ω_s	Spinning speed
ω_p	Precession speed
ω_n	Nutation speed
R	Beam length
θ	Nutation angle

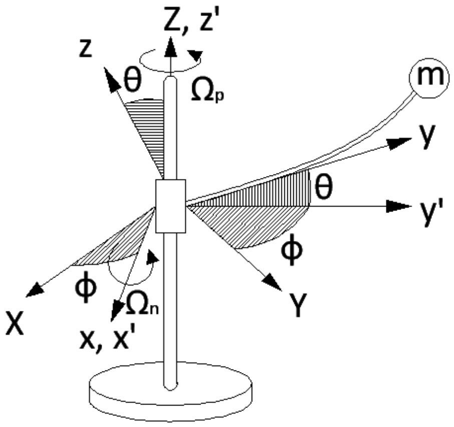

Figure 1. Motion of the beam about a point (spin is not shown in the figure to avoid complexity) [7]

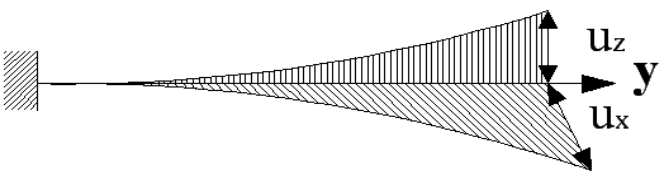

Figure 2. Deformation of the beam in x-y and y-z plane [7]

The expression of velocity and acceleration is obtained using Equations (3) and (4). The two transverse deflections of the tip mass are the two degrees of freedom of the system, as shown in Figure 2. The deformations along x-direction and z-direction are represented byu_x and $u_z$respectively. Deformation occurs only along x-direction and z-direction. The force equilibrium in x-direction and z-direction is represented in Equations (5) and (6). Table 1 lists the system parameters and their meanings.

$$\rho = u_x i + u_z k + Rj \tag{1}$$

$$\omega = \omega_n i + (\omega_p \cos\theta + \omega_s \sin\theta)k + (\omega_p \sin\theta + \omega_s \cos\theta)j \tag{2}$$

$$v = \frac{d\rho}{dt} + \omega \times \rho \tag{3}$$

$$a = \frac{dv}{dt} = \frac{\mathrm{d}^2\rho}{\mathrm{d}t^2} + \left(\omega \times (\omega \times \rho)\right) + 2(\omega \times \frac{d\rho}{dt}) \tag{4}$$

$$ma_x + \mathrm{ku_x} = 0 \tag{5}$$

$$ma_z + \mathrm{ku_z} = 0 \tag{6}$$

Governing Differential Equation of Motion

Using Equations (5) and (6) the following two differential equations are obtained as below – one for x-direction and the other for z-direction.

$$\ddot{u_x} + 2(\omega_p sin\theta + \omega_s cos\theta)\dot{u_z} + \left\{\frac{\mathrm{k}}{\mathrm{m}} - (\omega_p sin\theta + \omega_s cos\theta)^2 - (\omega_p cos\theta + \omega_s sin\theta)^2\right\}\mathrm{u_x} + \omega_\mathrm{n}(\omega_p cos\theta + \omega_s sin\theta)\mathrm{u_z} = 0 \tag{7}$$

$$\ddot{u_z} - 2(\omega_p sin\theta + \omega_s cos\theta)\dot{u_x} + \left\{\frac{\mathrm{k}}{\mathrm{m}} - (\omega_p sin\theta + \omega_s cos\theta)^2 - {\omega_\mathrm{n}}^2\right\}\mathrm{u_z} + \omega_\mathrm{n}(\omega_p cos\theta + \omega_s sin\theta)\mathrm{u_x} = 0 \tag{8}$$

The above equations are obtained without considering the centrifugal stiffening term. Young and Budynas (2002) introduced a stiffening term in the y-direction due to zero deformation along the longitudinal axis of the beam. The mathematical expression of centrifugal stiffness matrix is given as:

$$\{m\omega_n^2 + m(\omega_p \cos\theta + \omega_s \sin\theta)^2\}\begin{bmatrix}1.2 & 0\\ 0 & 1.2\end{bmatrix}$$

Incorporating the centrifugal stiffening term into the obtained differential equation of motion and rearranging the expressions in the form of a matrix equation, we have,

$$\begin{Bmatrix}\ddot{u_x}\\ \ddot{u_z}\\ \dot{u_x}\\ \dot{u_z}\end{Bmatrix} = \begin{bmatrix}0 & -2(\omega_p\sin\theta + \omega_s\cos\theta) & \varphi_1 & -\omega_\mathrm{n}(\omega_p\cos\theta + \omega_s\sin\theta)\\ 2(\omega_p\sin\theta + \omega_s\cos\theta) & 0 & -\omega_\mathrm{n}(\omega_p\cos\theta + \omega_s\sin\theta) & \varphi_2\\ 1 \quad 0 & & 0 \quad 0 & \\ 0 \quad 1 & & 0 \quad 0 & \end{bmatrix}\begin{Bmatrix}\dot{u_x}\\ \dot{u_z}\\ u_x\\ u_z\end{Bmatrix} \tag{9}$$

Where, $\boldsymbol{\varphi_1} = -\left(\frac{\mathbf{k}}{\mathbf{m}} + \mathbf{1.2}\boldsymbol{\omega_n^2} - \mathbf{0.4}\boldsymbol{\omega_p^2} - \mathbf{0.4}\boldsymbol{\omega_s^2}\right) + \mathbf{0.8\omega_s\omega_p sin2\theta} + \mathbf{0.6{\omega_p}^2cos2\theta} + \mathbf{0.6{\omega_s}^2cos2\theta}$

$$\boldsymbol{\varphi_2} = -\left(\frac{\mathbf{k}}{\mathbf{m}} + \mathbf{0.2}\boldsymbol{\omega_n^2} + \mathbf{0.1}\boldsymbol{\omega_p^2} + \mathbf{0.1}\boldsymbol{\omega_s^2}\right) - \mathbf{0.2\omega_s\omega_p sin2\theta} - \mathbf{1.1{\omega_p}^2cos2\theta} + \mathbf{1.1{\omega_s}^2cos2\theta}$$

Lemma

The principle of linear algebra states that if $\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{x}$ with $\boldsymbol{x}(\boldsymbol{t_0}) = \boldsymbol{x_0}$ and, $\boldsymbol{x}(\boldsymbol{t})$is the response and $\boldsymbol{x_0}$ is the initial condition, then,

$$\boldsymbol{x}(\boldsymbol{t}) = \mathbf{exp}[\int_{\boldsymbol{t_0}}^{\boldsymbol{t}} \boldsymbol{A}(\boldsymbol{\sigma})\boldsymbol{d\sigma}]\boldsymbol{x_0}$$

$$\boldsymbol{or}, \boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x_0} \sum_{\boldsymbol{k=0}}^{\infty} (\int_{\boldsymbol{t_0}}^{\boldsymbol{t}} \boldsymbol{A}(\boldsymbol{\sigma})\boldsymbol{d\sigma})^{\boldsymbol{k}}$$

Initial Conditions

The four sets of initial conditions are:

$$[1 \quad 0 \quad 0 \quad 0]^T, [0 \quad 1 \quad 0 \quad 0]^T, [0 \quad 0 \quad 1 \quad 0]^T \text{ and } [0 \quad 0 \quad 0 \quad 1]^T$$

RESULTS AND CONCLUSIONS

The system response, Equation (10) for different sets of initial conditions is developed and the specific cases are studied to give insight into the system dynamics.

$$\begin{Bmatrix} \dot{u_x} \\ \dot{u_z} \\ u_x \\ u_z \end{Bmatrix} = \begin{bmatrix} 1 & 2\frac{\omega_p}{\omega_n}(\cos\theta - 1) - 2\frac{\omega_s}{\omega_n}\sin\theta & \varphi_3 & -\omega_p\sin\theta - \omega_s(1-\cos\theta) \\ 2\frac{\omega_p}{\omega_n}(1-\cos\theta) + 2\frac{\omega_s}{\omega_n}\sin\theta & 1 & -\omega_s(1-\cos\theta) - \omega_p\sin\theta & \varphi_4 \\ t & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{u_{x0}} \\ \dot{u_{z0}} \\ u_{x0} \\ u_{z0} \end{Bmatrix} \tag{10}$$

Where, $\boldsymbol{\varphi_3} = -\left(\frac{\mathbf{k}}{\mathbf{m}} + \mathbf{1.2}\boldsymbol{\omega_n^2} - \mathbf{0.4}\boldsymbol{\omega_p^2} - \mathbf{0.4}\boldsymbol{\omega_s^2}\right)\mathbf{t} + \mathbf{0.4}\frac{\boldsymbol{\omega_p\omega_s}}{\boldsymbol{\omega_n}} - \mathbf{0.4}\frac{\boldsymbol{\omega_s\omega_p}}{\boldsymbol{\omega_n}}\mathbf{cos2\theta} + \mathbf{0.3}\frac{\boldsymbol{\omega_p}^2}{\boldsymbol{\omega_n}}\mathbf{sin2\theta} + \mathbf{0.3}\frac{\boldsymbol{\omega_s}^2}{\boldsymbol{\omega_n}}\mathbf{sin2\theta}$

$$\varphi_4 = -\left(\frac{k}{m} + 0.2\omega_n^2 + 0.1\omega_p^2 + 0.1\omega_s^2\right)t - 0.1\frac{\omega_p\omega_s}{\omega_n} + 0.1\frac{\omega_s\omega_p}{\omega_n}\cos 2\theta - 0.55\frac{\omega_p^2}{\omega_n}\sin 2\theta + 0.55\frac{\omega_s^2}{\omega_n}\sin 2\theta$$

Simultaneously precessing, spinning and nutating

The response for unit velocity in x-direction as the initial condition is below:

$$u = \begin{cases} 1, & in x - direction \\ 2\frac{\omega_p}{\omega_n}(1 - cos\theta) + 2\frac{\omega_s}{\omega_n}sin\theta, & in z - direction \end{cases}$$

The response for unit velocity in z-direction as the initial condition is below:

$$u = \begin{cases} -2\frac{\omega_p}{\omega_n}(1 - cos\theta) - 2\frac{\omega_s}{\omega_n}sin\theta, & in x - direction \\ 1, & in z - direction \end{cases}$$

Figure 3 shows the response versus time for unit velocities as the initial conditions, which is a sinusoidal response.

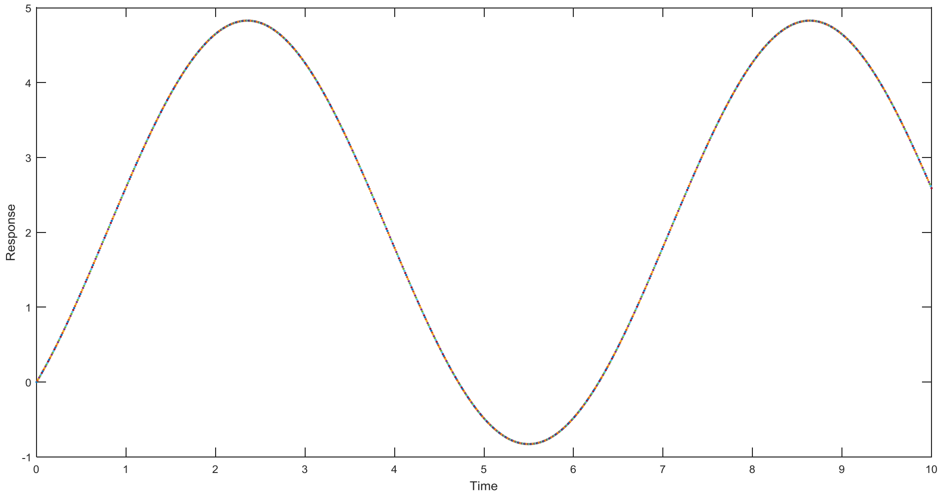

Figure 3. Response versus time for unit velocity as initial condition

The response for displacement in x-direction as the initial condition is below:

$$u = \begin{cases} -\left(\frac{k}{m} + 1.2\omega_n^2 - 0.4\omega_s^2 - 0.4\omega_p^2\right)t + f(\omega_p, \omega_n, \omega_s, t, cos\theta, sin\theta), & inx - direction \\ -\omega_s(1 - cos\theta) - \omega_p sin\theta, & inz - direction \end{cases}$$

The response for unit velocity in x-direction as the initial condition is below:

$$u = \begin{cases} -\omega_s(1 - cos\theta) - \omega_p sin\theta, & inx - direction \\ -\left(\frac{k}{m} + 0.2\omega_n^2 + 0.1\omega_s^2 + 0.1\omega_p^2\right)t + f(\omega_p, \omega_n, \omega_s, t, cos\theta, sin\theta), & inz - direction \end{cases}$$

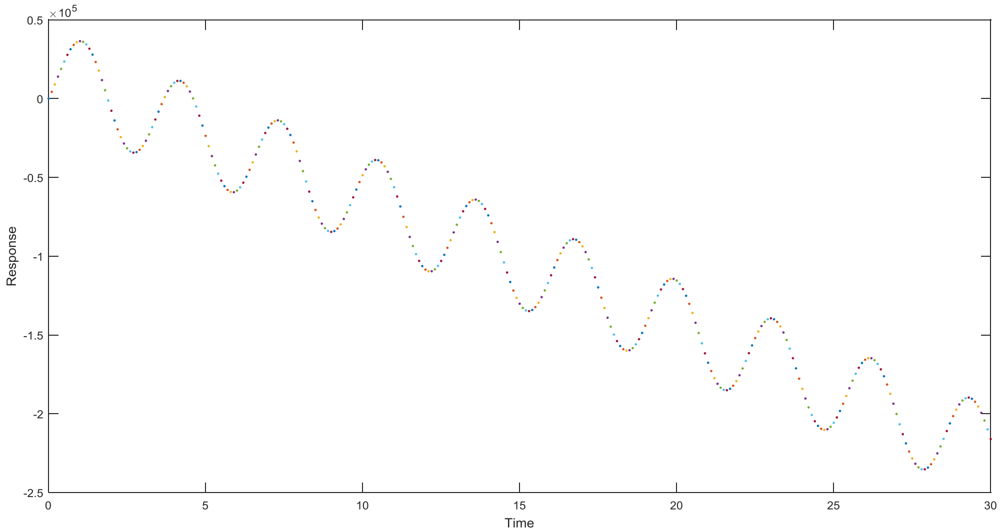

Figure 4. Response versus time for unit displacement as initial condition

Figure 4 shows the response versus time for unit velocities as the initial conditions.
Simultaneously precessing and spinning

Considering no nutation, the system response (represented by$u$ below) is a linear function of time for unit velocities as the initial condition. Thus, the system is unbounded. This instability increases with time. The variation is established below: ($\theta_n$is the angle of nutation – fixed for this case)

$$u = t(\omega_s cos\theta_n + \omega_p sin\theta_n)$$

But for unit displacements as the initial condition, the response is stable as the stiffness of the system dominates. Due to high system stiffness to mass ratio, the response is always negative.

$$u = -\left(\frac{k}{m} + 0.1\omega_p^2 + 0.1\omega_s^2\right) t + f(\omega_p, cos\theta_n, sin\theta_n, \omega_s, t,)$$

Simultaneously Nutating and Spinning

Considering no precession, the system response for the unit velocities as initial condition is bounded. It is sinusoidal function of nutation angle at any instant.

$$u = \frac{\omega_s}{\omega_n} sin\theta$$

Thus, it is clear from the above expression that, with increase in spinning speed, the response magnifies, but, with increase in nutation speed, the response becomes more bounded and vice-versa. For unit displacements as the initial condition, the response is negative since stiffness dominates the other parameters.

Simultaneously precessing and nutating

Considering no spinning motion, the response for unit velocities as the initial condition is either a product of a trigonometric function of time and precession speed, or, a function of system stiffness as shown below:

$$u = -\left(\frac{k}{m} + 0.2\omega_n^2 + 0.1\omega_p^2\right) t + f(\omega_p, cos\theta_n, sin\theta_n, \omega_s, t,)$$

$$u = \omega_p sin\theta$$

For the first expression, the system stability is ensured since the system stiffness dominates the equation of motion. For the second expression, the system is bounded and stable. With increase in precession speed, the response becomes unbounded. For unit displacements as the initial condition, the response is shown below:

$$u = \frac{\omega_p}{\omega_n}(1 - cos\theta)$$

It is clear from the above expression that, the response becomes stable with increase in nutation speed and decrease in precession speed and vice-versa.

Only spinning

For unit velocities as the initial condition, the response is a product of spinning speed, time and a trigonometric function of nutation angle $\theta_n$as shown:

$$u = \omega_s t cos\theta_n$$

Thus, the response is a linear function of time, hence, system is unstable. Also, with increase in spinning speed, the system becomes unbounded. But, for unit displacements as the initial condition, the system response is negative and always stable since system stiffness dominates the stability criterion.

Only precessing

For unit velocities as the initial condition, the response is shown above to be a linear function of time; hence, system is unstable. Also, with increase in spinning speed, the system becomes unbounded.

$$u = \omega_p t sin\theta_n$$

But, for unit displacements as the initial condition, the system response is negative and always stable since system stiffness dominates the stability criterion.

Only nutating

For unit velocities as the initial condition, the response is a constant. Hence, the system is stable and bounded. For unit displacements as the initial condition, the system response is always negative and stable as shown below:

$$u = -\left(\frac{k}{m} + 0.2\omega_n^2\right)t$$

The entire analysis is incomplete without the consideration of mass of the beam into account. This assumption might not always provide accurate results. Also, the effect of angular accelerations in nutation, precession and spinning directions is not taken into consideration to avoid complexity. Also, following the lemma proposed from theory of linear algebra, only the first two terms are used for analysis. Higher terms would have given more accurate results. All these shortcomings shall be taken care of in our next endeavor.

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