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Gradient Ricci-Bourguignon Solitons on Static Spacetime

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Abstract

In this paper, we investigate gradient Ricci-Bourguignon solitons on static spacetimes from both geometric and physical perspectives. We first obtain necessary conditions under which a static spacetime admits a gradient Ricci-Bourguignon soliton and show that the induced structure on the base manifold leads naturally to a gradient Ricci-Bourguignon soliton. In the vacuum static case, we prove that the soliton necessarily becomes steady. For non-compact and connected static spacetimes, it is shown that an expanding gradient Ricci-Bourguignon soliton forces the warping function to satisfy a Schr*ö*dinger-type equation. Further, we analyze static perfect fluid spacetimes admitting gradient Ricci-Bourguignon solitons and derive explicit bounds on the soliton constant ensuring the null convergence and strong energy conditions. It is also proved that when the potential function coincides with the warping function, the base manifold becomes Einstein. Additional results are established for Ricci symmetric and weakly Ricci symmetric base manifolds, highlighting the influence of the Bourguignon parameter. Finally, we characterize the nature of gradient Ricci-Bourguignon solitons on four-dimensional half conformally flat base manifolds, particularly in the stiff matter case.

Keywords: Static spacetime, Gradient Ricci–Bourguignon solitons, Conharmonic curvature tensor, Perfect fluid spacetime, Shr*ö*dinger's equation, Ricci symmetric spacetime, Stiff matter.

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1. INTRODUCTION

Ricci solitons arise naturally as self-similar solutions to the Ricci flow and serve as important models in the study of singularity formation and geometric evolution. Their genesis is classically attributed to Hamilton's seminal work on the Ricci flow [21]. A Riemannian metric g on a smooth manifold M together with a vector field V and a constant λ is called a *Ricci soliton* if

$$\mathcal{L}_V g + 2\,\mathrm{Ric} + 2\lambda\, g = 0, \tag{1.1}$$

where $\mathcal{L}_V$ denotes the Lie-derivative with respect to V. Depending on the sign of λ, Ricci solitons are classified as shrinking $(\lambda < 0)$, steady $(\lambda = 0)$, or expanding $(\lambda > 0)$ [21]. These structures have important connections to general relativity, cosmology and string theory, where the Ricci flow describes the renormalization of nonlinear sigma models and the soliton is called *gradient* when $V = \nabla f$ for some smooth function f (the potential) so that the equation becomes (1.1)

$$\mathrm{Hess}\, f + \mathrm{Ric} + \lambda\, g = 0, \tag{1.2}$$

where $Hess\ \ f$ denotes the Hessian of a smooth function f. Ricci solitons have been widely studied both for their intrinsic geometric interest and for applications to mathematical relativity and geometric analysis.

The Ricci-Bourguignon flow is a natural one-parameter family of modifications of the Ricci flow that include the scalar curvature term. It was introduced as an extension of classical geometric flows and motivated several natural generalizations of Ricci solitons. A Ricci-Bourguignon soliton is a metric-vector-scalar triple (g, ξ, λ) satisfying the equation [6]

$$\mathcal{L}_\xi g + 2\,\mathrm{Ric} - 2\rho\, r\, g + 2\lambda\, g = 0, \tag{1.3}$$

for real parameter ρ, scalar curvature r and soliton constant λ. When $\xi = \nabla f$ one obtains the gradient Ricci–Bourguignon soliton whose equation is given by

$$\mathrm{Hess}\, f + \mathrm{Ric} - \rho\, r\, g + \lambda\, g = 0, \tag{1.4}$$

where Ric is the Ricci curvature tensor of the metric g and r represents the scalar curvature of M. The constant ρ is a real parameter associated with the Ricci-Bourguignon structure, while λ is a real constant.

Catino [7] investigated the Ricci–Bourguignon flow and established the short-time existence and uniqueness of solutions starting from an initial Riemannian metric. Subsequently, in 2021, Siddiqui and Siddiqui [30] constructed an explicit example of a four-dimensional spacetime admitting a Ricci-Bourguignon soliton. Related investigations were carried out by S. Azami [2] in the context of h-Ricci-Bourguignon solitons. After that R. Mi [26] proved that under suitable assumptions, a complete non-compact gradient Ricci-Bourguignon soliton possesses non-negative scalar curvature. Later, Blaga and Taştan [4] obtained several rigidity results and derived important curvature properties of Ricci- Bourguignon solitons. Dhriti et al. [19], studied gradient Ricci-Bourguignon solitons on paracontact metric manifolds and showed that under certain conditions, such solitons reduce to Einstein manifolds. Although the Ricci– Bourguignon flow was introduced earlier than the Ricci flow, the notion of Ricci–Bourguignon solitons emerged subsequently. Catino et al. [8] examined rigidity aspects of ρ-Einstein solitons and demonstrated the constancy of scalar curvature under appropriate hypotheses; here, the ρ-Einstein soliton coincides with the Ricci-Bourguignon soliton. Furthermore, in 2016, Catino et al. [9] extended their earlier work [8] and showed that Ricci-Bourguignon solitons exhibit stronger rigidity properties compared to Ricci solitons. More recently, Chen et al. [12] analyzed Ricci-Bourguignon solitons on statistical submersions with parallel vertical or horizontal distributions, as well as those admitting conformal or gradient potential vector fields. Finally, in 2021, S. Dwivedi [20] formally introduced the definition of Ricci-Bourguignon solitons as a generalization of Ricci solitons and also established an isometric correspondence between the Euclidean sphere and compact Ricci-Bourguignon almost solitons under suitable conditions.

An n-dimensional semi-Riemannian manifold endowed with a metric of signature $(1, n-1)$ is called a Lorentzian manifold of dimension n [27]. In the framework of general relativity, a spacetime is modeled as a connected four-dimensional Lorentzian manifold equipped with a Lorentzian metric of signature $(-, +, +, +)$. Over the years, significant attention has been devoted to the investigation of physical and geometric properties of spacetimes admitting various geometric structures. Since static spacetimes generalize static vacuum solutions, they form an important class of manifolds of interest to both mathematicians and physicists.

In this context, Kobayashi and Obata [23] studied static spacetimes on Lorentzian manifolds satisfying the Einstein field equations with perfect fluid matter content and analyzed the conformal flatness of such spacetimes. Conditions involving the Riemannian factor and the warping function of standard static spacetimes were obtained in [14]. Moreover, static perfect fluid spacetimes on generalized Robertson-Walker spacetimes were investigated in [17], where it was shown that, under suitable restrictions, such spacetimes belong to Petrov types I, D or O. Several authors have further examined static perfect fluid spacetimes in different geometric settings, including contact metric manifolds [11], compact manifolds [13], half conformally flat manifolds [24] and paracontact metric manifolds [28]. In addition, Ricci solitons on spherically symmetric static spacetimes were studied in [34], where it was shown that certain classes of such metrics admit shrinking, steady, or expanding Ricci solitons. The geometry of standard static spacetimes in the setting of almost Ricci-Yamabe solitons with conformal vector fields was further analyzed in [32].

After the development of spaces of constant curvature, the classification of locally symmetric Riemannian manifolds became a central topic in differential geometry. The study of locally symmetric structures has also been extended to perfect fluid spacetimes. In particular, Mallick and De [25] derived conditions for the existence of perfect fluid semi-Riemannian symmetric spacetimes. Related investigations on pseudo-symmetric manifolds [10] and weakly symmetric manifolds [35] have also been carried out. These developments motivated extensive research on various types of Ricci solitons in spacetime geometry, including Ricci solitons [33], η-Ricci-Bourguignon solitons [18], Ricci-Yamabe solitons [31] and conformal Ricci solitons [29].

Very recently, Yadav and Saxena [36] investigated static spacetimes admitting almost gradient Ricci solitons and derives several geometric and physical consequences. The authors show that such solitons reduce to steady gradient Ricci solitons in vacuum static spacetimes and obtain conditions related to energy inequalities in static perfect fluid settings. Rigidity results are established for Ricci symmetric, weakly Ricci symmetric, and half conformally flat base manifolds, with explicit characterization of the soliton constant, particularly in the stiff matter case. The work provides a meaningful extension of Ricci soliton theory to static and perfect fluid spacetimes.

Motivated by the above considerations, in this paper we investigate gradient Ricci-Bourguignon solitons on static spacetimes from both geometric and physical viewpoints. We derive necessary conditions under which a static spacetime admits a gradient Ricci-Bourguignon soliton and show that the induced structure on the base manifold naturally inherits a corresponding soliton structure. Special attention is given to vacuum and non-compact static spacetimes, where rigidity results and Schr$\ddot{o}$dinger-type equations for the warping function are obtained. Furthermore, we study static perfect fluid spacetimes admitting such solitons and establish bounds on the soliton constant ensuring the null convergence and strong energy conditions. Additional results are obtained for Ricci symmetric and weakly Ricci symmetric base manifolds and a detailed characterization is provided for four-dimensional half conharmonically flat cases, particularly in the stiff matter setting.

2. PRELIMINARIES

An $(n+1)$-dimensional Lorentzian warped product manifold $\tilde{M}^{(n+1)} = M^n \times_f \mathbb{R}$ is called a static spacetime [23] if it is equipped with the metric

$$\tilde{g} = g - f^2 dt^2, \tag{2.1}$$

where (M^n, g) is an n-dimensional Riemannian manifold and $f > 0$ is a smooth function defined on M.

We consider the Einstein field equations on $(\tilde{M}, \tilde{g})$ with perfect fluid as the matter source, given by

$$\tilde{S} - \frac{\tilde{r}}{2}\tilde{g} = -\rho g - \sigma f^2 dt^2, \tag{2.2}$$

where $\tilde{S}$ and $\tilde{r}$ denote respectively the Ricci tensor and scalar curvature of $(\tilde{M}, \tilde{g})$, while ρ and σ are smooth functions on $\tilde{M}$ interpreted as isotropic pressure and energy density.

From equations (2.1) and (2.2), the static perfect fluid spacetime condition is equivalent to

$$f\left(S - \frac{r}{n}g\right) = H^f - \frac{\Delta f}{n}g, \tag{2.3}$$

and

$$\Delta f = \left(\frac{n-2}{2(n-1)}r + \frac{n}{n-1}\rho\right)f, \tag{2.4}$$

where S, r and Δ denote respectively the Ricci tensor, scalar curvature and Laplacian on (M^n, g), and H^f is the Hessian of f.

In particular, if $\rho = r = 0$, equations (2.3) and (2.4) reduce to the static vacuum Einstein equations

$$fS = H^f, \qquad \Delta f = 0. \tag{2.5}$$

Moreover, setting $\rho = -\frac{r}{2}$ in (2.4) and using it in (2.3), one obtains the Fischer–Marsden equation

$$H^f - fS - \Delta f g = 0. \tag{2.6}$$

Let ∇ be the Levi-Civita connection on (M^n, g). The Levi-Civita connection $\tilde{\nabla}$ on $(\tilde{M}, \tilde{g})$ is determined by [32]

$$\tilde{\nabla}_{\partial_t}\partial_t = f\mathrm{grad}f, \quad \tilde{\nabla}_{\partial_t}X = \tilde{\nabla}_X\partial_t = X(\ln f)\partial_t, \quad \tilde{\nabla}_X Y = \nabla_X Y, \tag{2.7}$$

for all vector fields X, Y tangent to M.

The curvature tensors of $(\tilde{M}, \tilde{g})$ are related to those of (M, g) by

$$\tilde{R}(X, \partial_t)\partial_t = -f\nabla_X \mathrm{grad}f, \tag{2.8}$$

$$\tilde{R}(\partial_t, \partial_t)\partial_t = \tilde{R}(\partial_t, \partial_t)X = \tilde{R}(X, Y)\partial_t = 0, \tag{2.9}$$

$$\tilde{R}(\partial_t, X)Y = \frac{1}{f}H^f(X, Y)\partial_t, \tag{2.10}$$

$$\tilde{R}(X, Y)Z = R(X, Y)Z, \tag{2.11}$$

where R is the curvature tensor of (M, g).

The Ricci tensor and scalar curvature of $\tilde{g}$ are given by

$$\tilde{S}(\partial_t, \partial_t) = f\Delta f, \tag{2.12}$$

$$\tilde{S}(X, \partial_t) = 0, \tag{2.13}$$

$$\tilde{S}(X, Y = S(X, Y) - \frac{1}{f}H^f(X, Y), \tag{2.14}$$

$$\tilde{r} = r - \frac{2}{f}\Delta f \tag{2.15}$$

.

A non-flat Riemannian manifold (M, g) of dimension $n > 2$ is said to be weakly Ricci symmetric [35] if its Ricci tensor S satisfies

$$(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(Y)S(Z, X) + \gamma(Z)S(X, Y), \tag{2.16}$$

for all vector fields X, Y, Z, where α, β, γ are non-zero 1-forms. On such a manifold, one defines the 1-form δ by

$$\delta(X) = \beta(X) - \gamma(X).$$

Lemma 2.1. *[15] If $\delta \neq 0$ on a weakly Ricci symmetric manifold (M, g), then the Ricci tensor has the form*

$$S(X, Y) = -r\mu(X)\mu(Y), \tag{2.17}$$

where r is the scalar curvature and μ is a non-zero 1-form defined by

$$\mu(X) = g(X, \rho), \tag{2.18}$$

with ρ denoting the associated basic vector field.

Lemma 2.2. *[15] In a 4-dimensional weakly Ricci symmetric spacetime, the scalar curvature cannot vanish, since $r = 0$ provided $S = 0$ from equation (2.2), which contradicts the definition of $(wRs)_4$.*

Definition 2.3. *[27] An n-dimensional Lorentzian manifold is called a perfect fluid spacetime if its Ricci tensor satisfies*

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y), \tag{2.19}$$

for all vector fields X, Y, where α and β are scalar functions and $\eta(X) = g(X,\rho)$ with ρ a unit timelike vector field.

Definition 2.4. *A vector field ξ on a Riemannian manifold (M,g) is called torse-forming if*

$$\nabla_X\xi = fX + \nu(X)\xi, \tag{2.20}$$

for all vector fields X on M, where f is a smooth function and ν is a 1-form.

Lemma 2.5. *[5] Let (M^n,g) be an n-dimensional Riemannian manifold admitting a gradient Ricci-Bourguignon soliton then the following identities hold:*

1. $\Delta f + (1 - n\rho)\, r + n\lambda = 0,$
2. $\nabla\Delta f + (1 - n\rho)\nabla r + n\nabla\lambda = 0$
3. $\nabla\Delta f + \mathrm{Ric}(\nabla f) + \frac{1}{2}(1 - 2\rho)\nabla r + \nabla\lambda = 0$
4. $R(X,Y,Z,\nabla f) = d(\lambda)(Y)g(X,Z) - d(\lambda)(X)g(Y,Z)$
   $+(1-\rho)[(\nabla_Y S)(X,Z) - (\nabla_X S)(Y,Z)],$

for all smooth vector fields X, Y, Z on M.

The conharmonic curvature tensor [22] K on (M,g) is defined as follows:

$$\begin{aligned} K(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}\{&g(Y,Z)QX \\ - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\},& \end{aligned} \tag{2.21}$$

and the cotton tensor is given by

$$\begin{aligned} \mathcal{C}^k(X,Y)Z &= (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\ &- \frac{1}{2(n-1)}\{g(Y,Z)dr(X) - g(X,Z)dr(Y)\},. \end{aligned} \tag{2.22}$$

for all smooth vector fields $X,\ ,Y,\ Z$ on M, where Q is the Ricci operator. With the help of equations (2.11) and (2) and Lemma (2.5), the conharmonic curvature tensor can be expressed as follows:

Lemma 2.6. *[5] If an n-dimensional manifold (M, g) is admitting a gradient Ricci-Bourguignon soliton then*

$$\begin{aligned}
&K(X,Y,Z,\nabla f) = (\rho - \frac{1}{2(n-1)})T(X,Y,Z) - (\frac{n-2}{n-1})\mathcal{C}^k(X,Y)Z \\
&- \frac{1}{(n-2)}[g(Y,Z)S(X,\nabla f) - g(X,Z)S(Y,\nabla f) + S(Y,Z)g(X,\nabla f) \\
&- S(X,Z)g(Y,\nabla f)],
\end{aligned} \tag{2.23}$$

where tensor T appearing in equation (2.23) is defined by

$$T(X,Y,Z) = g(Y,Z)g(X,\nabla f) - g(X,Z)g(Y,\nabla f), \tag{2.24}$$

for all smooth vector fields X, Y, Z on M.

Thus we can state some necessary propositions:

Proposition 2.1. [1] Let $M = (a,b)_f \times H$ be a standard static spacetime with (H,h) a Ricci flat Riemannian manifold of dimension $H \geq 2$. Then, $h(w,w)\Delta f - Hess(f)(w,w) - (\frac{n-2}{n-1})\mathcal{C}^k(w,\nabla f,w) + (\rho - \frac{1}{2(n-1)})T(w,\nabla f,w) \geq 0, \forall\, w \in TH$ if and only if M satisfies the null convergence condition.

Proposition 2.1. [1] Let $M = (a,b)_f \times H$ be a standard static spacetime with (H,h) Ricci flat of dimension $H \geq 2$. Then M satisfies the null convergence condition if and only if it satisfies the strong energy condition and the identity:

$$\begin{aligned}
(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \rho\{g(Y,Z)dr(X) \\
- g(X,Z)dr(Y)\} + (\frac{n-2}{n-1})\mathcal{C}^k(X,Y)Z.
\end{aligned}$$

3. GRADIENT RICCI-BOURGUIGNON SOLITONS ON STATIC SPACETIME

In this section, we investigate a gradient Ricci-Bourguignon soliton structure on a static spacetime $(\tilde{M}^{n+1}, \tilde{g})$, where the potential function is denoted by $\tilde{\phi}$ and the soliton constant by $\tilde{\lambda}$. The gradient Ricci-Bourguignon soliton on $(\tilde{M}^{n+1}, \tilde{g})$ is defined by

$$H^{\tilde{\phi}}(\tilde{X},\tilde{Y}) + \tilde{S}(\tilde{X},\tilde{Y}) - \rho\,\tilde{r}\,\tilde{g}(\tilde{X},\tilde{Y}) + \tilde{\lambda}\,\tilde{g}(\tilde{X},\tilde{Y}) = 0, \tag{3.1}$$

for all smooth vector fields $\tilde{X}, \tilde{Y}$ on $\tilde{M}$, where $H^{\tilde{\phi}}$ denotes the Hessian of the smooth function $\tilde{\phi}$, $\tilde{S}$ and $\tilde{r}$ are respectively the Ricci tensor and scalar curvature of $\tilde{g}$, and ρ is the Bourguignon parameter.

Theorem 3.1. *Let $\tilde{M} = M \times_f \mathcal{R}$ be a static spacetime with metric $\tilde{g} = g - f^2dt^2$. If $(\tilde{g}, \tilde{\lambda}, \tilde{\phi})$ is an gradient Ricci-Bourguignon soliton on $\tilde{M}$ and $H^f = 0$ with $(\frac{\Delta f}{f})$ is constant then (g, λ, ϕ) is gradient Ricci-Bourguignon soliton on M.*

Proof. Let $(\tilde{M}^{n+1}, \tilde{g})$ be a static spacetime and admits a gradient Ricci-Bourguignon soliton. Then by the help of equation (3.1) at (X, Y) on M, we get

$$H^{\tilde{\phi}}(X,Y) + \tilde{S}(X,Y) - \rho\,\tilde{r}\,\tilde{g}(X,Y) + \tilde{\lambda}\,\tilde{g}(X,Y) = 0, \tag{3.2}$$

for all smooth vector fields X, Y on M. It is known that [?], $H^{\tilde{\phi}}(X,Y) = H^{\phi}(X,Y)$, where $\phi = \tilde{\phi}|_M$ and using equations (2.1) and (2.4) in above equation, we have

$$\begin{aligned} &H^{\phi}(X,Y) + S(X,Y) - \rho r g(X,Y) + \tilde{\lambda} g(X,Y) \\ &= (\frac{1}{f})H^f(X,Y) - (\frac{2\rho}{f})(\Delta f)g(X,Y). \end{aligned} \tag{3.3}$$

Suppose that $H^f = 0$, we have

$$H^{\phi}(X,Y) + S(X,Y) - \rho r g(X,Y) + [\tilde{\lambda} + (\frac{2\rho}{f})(\Delta f)]g(X,Y) = 0. \tag{3.4}$$

If $(\frac{\Delta f}{f})$ is constant such that $\lambda = [\tilde{\lambda} + \frac{2\rho}{f}(\Delta f)]$ on M then, equation (3.4) yields

$$H^{\phi}(X,Y) + S(X,Y) - \rho r g(X,Y) + \lambda g(X,Y) = 0. \tag{3.5}$$

Hence the theorem. □

Theorem 3.2. *If a static spacetime $(\tilde{M}, \tilde{g})$ admits a Ricci-Bourguignon soliton, then the base manifold (M, g) naturally admits a gradient η-Ricci-Bourguignon soliton, where the η-term is generated by the warping function via $\eta = d(ln\ f)$ and $\psi = \phi - ln\ f$.*

Proof. Using the fact that [3]

$$\frac{1}{f}H^f = H^{ln\,f} + \frac{1}{f^2}df \otimes df, \tag{3.6}$$

with the help of equation (3.3), we have

$$\begin{aligned} &H^{\phi}(X,Y) + S(X,Y) - \rho r g(X,Y) + \lambda g(X,Y) \\ &- H^{ln\,f} + \frac{1}{f^2}df \otimes df - (\frac{2\rho}{f})(\Delta f)g(X,Y). \end{aligned} \tag{3.7}$$

where $\mu = 1$ and $\lambda = \tilde{\lambda} + \frac{2\rho}{f}\Delta f$.
Now taking,

$$H^{\psi} = H^{\phi} - H^{ln\,f}. \tag{3.8}$$

Let us define $ln\ f = l$ and using $\eta = d(ln\ f)$ equation (3.7), becomes

$$H^{\psi}(X,Y) + S(X,Y) - \rho r g(X,Y) + \lambda_m g(X,Y) - \eta(X)\eta(Y), \tag{3.9}$$

□

where $\lambda_m = (\lambda + \frac{2\rho}{f}\Delta f)$ and $\mu = 1$. Hence $(g, \psi, \lambda_m, \mu)$ is a gradient η-Ricci-Bourguignon soliton on M.

Theorem 3.3. *Let $\tilde{M} = M \times_f \mathcal{R}$ be a vacuum static spacetime with metric $\tilde{g} = g - f^2 dt^2$. If $(\tilde{g}, \tilde{\lambda}, \tilde{\phi})$ is a gradient Ricci-Bourguignon soliton on $\tilde{M}$ and $H^{\tilde{\phi}}(\partial_t, \partial_t) = 0$, then the gradient Ricci-Bourguignon soliton is steady.*

Proof. Let $(\tilde{M}^{n+1}, \tilde{g})$ be a static spacetime. Then from equation (3.1), we have

$$H^{\tilde{\phi}}(\partial_t, \partial_t) + \tilde{S}(\partial_t, \partial_t) - \rho \tilde{r} \tilde{g}(\partial_t, \partial_t) + \tilde{\lambda} \tilde{g}(\partial_t, \partial_t) = 0. \tag{3.10}$$

By using equation (2.1), we $\tilde{g}(\partial_t, \partial_t) = -f^2$, thus equation (3.10) reduces to

$$\tilde{\lambda} = \rho \tilde{r} + \frac{1}{f^2}\{H^{\tilde{\phi}}(\partial_t, \partial_t) + \tilde{S}(\partial_t, \partial_t)\}. \tag{3.11}$$

Using equation (2) in equation (3.11) with the assumption of $H^{\tilde{\phi}}(\partial_t, \partial_t) = 0$, we have

$$\tilde{\lambda} = \rho \tilde{r} + \frac{\Delta f}{f}, \tag{3.12}$$

for a vacuum static spacetime, we have $\Delta f = 0$ and $\tilde{r} = 0$ (under vacuum Einstein equations) therefore, we have

$$\tilde{\lambda} = 0. \tag{3.13}$$

Thus the soliton is steady. □

Theorem 3.4. *Let $\tilde{M} = M \times_f \mathcal{R}$ be a connected and non-compact static spacetime admitting a gradient Ricci-Bourguignon soliton on $\tilde{M}$ and $H^{\tilde{\phi}}(\partial_t, \partial_t) = 0$, then the warping function f is a solution of Schrödinger's equation on the base manifold M.*

Proof. Since (g, λ, ϕ) is a gradient Ricci-Bourguignon soliton on M, thus from equation (3.12), we have

$$\tilde{\lambda} = \rho \tilde{r} + \frac{\Delta f}{f}, \tag{3.14}$$

which implies

$$(\Delta - V)f = 0, \tag{3.15}$$

where $V = \tilde{\lambda} - \rho \tilde{r}$ is the Schrödinger's potential which depends upon $\tilde{\lambda}$ and $\tilde{r}$. Hence the theorem. □

4. GRADIENT RICCI-BOURGUIGNON SOLITONS ON STATIC PERFECT FLUID SPACETIME

In this section, we study gradient Ricci-Bourguignon solitons on static perfect fluid spacetime $(\tilde{M}^{n+1}, \tilde{g})$ wirh potential function as $\tilde{\phi}$ and soliton constant $\tilde{\lambda}$.

Theorem 4.1. *Let* $\tilde{M} = M \times_f \mathcal{R}$ *be a static perfect fluid spacetime with metric* $\tilde{g} = g - f^2dt^2$*. If* $(\tilde{g}, \tilde{\lambda}, \tilde{\phi})$ *is a gradient Ricci-Bourguignon soliton on* $\tilde{M}$ *with* $H^{\tilde{\phi}}(\partial_t, \partial_t) = 0$*, then soliton constant*

$$\tilde{\lambda} = \rho\tilde{r} + \sigma.$$

Proof. Let $\tilde{M} = M \times_f \mathcal{R}$ be a static perfect fluid spacetime admitting gradient Ricci-Bourguignon soliton, then we have

$$H^{\tilde{\phi}} + (\sigma + p)\eta \otimes \eta + (p - \rho\tilde{r} + \tilde{\lambda})\tilde{g} = 0, \tag{4.1}$$

where p is isotropic pressure of the perfect fluid spacetime and σ is the energy density. Using $\tilde{g}(\partial_t, \partial_t) = -f^2$ and $\eta(\partial_t) = -f$ with the assumption $H^{\tilde{\phi}}(\partial_t, \partial_t) = 0$, we have

$$(\sigma + \rho\tilde{r} - \tilde{\lambda})f^2 = 0, \tag{4.2}$$

since $f^2 \neq 0$, so , we have

$$\tilde{\lambda} = \rho\tilde{r} + \sigma,$$

which completes the proof. □

Theorem 4.2. *Let* $\tilde{M} = M \times_f \mathcal{R}$ *be a static perfect fluid spacetime with* (M, g) *is Ricci flat manifold and* $dim \geq 2$ *admitting gradient Ricci-Bourguignon soliton with* $H^{\tilde{\phi}} = 0$*, where* $\phi = \tilde{\phi}|_M$*, then* $\tilde{M}$ *satisfies the null convergence condition if*

$$\tilde{\lambda} \leq \frac{2(1-\rho)}{(n-1)}\frac{\Delta f}{f} - \frac{(1-\rho)}{(n-1)}r.$$

Proof. Let $(\tilde{g}, \tilde{\lambda}, \tilde{\phi})$ is gradient Ricci-Bourguignon soliton on static perfect fluid spacetime $\tilde{M} = M \times_f \mathcal{R}$. Then with the help of equation (2.3), we have

$$g(X,X)\Delta f - H^f(X,X) = (n-1)H^{\phi}(X,X) - fn\tilde{S}(X,X) + f\tilde{r}g(X,X). \tag{4.3}$$

Using equation (3.1) in above equation, we get

$$g(X,X)\Delta f - H^f(X,X) = f(n-1)H^{\phi}(X,X) - f[\tilde{\lambda} + (1-\rho)\tilde{r}]g(X,X). \tag{4.4}$$

Suppose that $H^{\phi} = 0$, then from equation (4.4), we have

$$g(X,X)\Delta f - H^{f}(X,X) = f[\tilde{\lambda} + (1-\rho)\tilde{r}]g(X,X). \tag{4.5}$$

Now assume that $g(X,X)\Delta f - H^{f}(X,X) \geq 0$, then from equation (4.5) yields

$$\tilde{\lambda} \leq \frac{2(1-\rho)}{(n-1)}\frac{\Delta f}{f} - \frac{(1-\rho)}{(n-1)}r. \tag{4.6}$$

Consequently, applying proposition 2.1 to equation (4.6) yields the stated result. □

By applying proposition 2.2 in conjunction with theorem 4.2, we obtain the following result.

Theorem 4.3. *Let $\tilde{M} = M \times_f \mathcal{R}$ be a static perfect fluid spacetime with (M,g) is Ricci flat manifold and $dim \ \geq 2$ admitting gradient Ricci-Bourguignon soliton with $H^{\tilde{\phi}} = 0$, where $\phi = \tilde{\phi}|_M$, then $\tilde{M}$ satisfies the strong energy condition if*

$$\tilde{\lambda} \leq \frac{(1-\rho)}{(n-1)}r - \frac{2(1-\rho)}{(n-1)}\frac{\Delta f}{f}.$$

5. GRADIENT RICCI-BOURGUIGNON SOLITON ON BASE MANIFOLD OF STATIC PERFECT FLUID SPACETIME

In this section, we study the gradient Ricci–Bourguignon soliton structure on the base manifold M of a static perfect fluid spacetime $\tilde{M} = M \times_f \mathcal{R}$. We assume that the potential function coincides with the warping function f and the soliton constant is λ. The gradient Ricci-Bourguignon soliton on M is defined by

$$H^{f}(X,Y) + S(X,Y) - \rho\, r\, g(X,Y) + \lambda\, g(X,Y) = 0, \tag{5.1}$$

for all smooth vector fields X, Y on M, where H^{f} denotes the Hessian of the smooth function f, S is the Ricci tensor of the metric g, r is the scalar curvature of M, and ρ and λ are smooth functions on M.

Theorem 5.1. *Let M be the base manifold of a static perfect fluid spacetime $\tilde{M} = M \times_f \mathcal{R}$. If M admits a gradient Ricci-Bourguignon soliton with potential function coinciding with the warping function f, then M is an Einstein manifold.*

Proof. Let (M^{n}, g) be the base manifold of a static perfect fluid spacetime $(\tilde{M}^{n+1}, \tilde{g})$ admitting gradient Ricci-Bourguignon soliton. Using equation (2.3) in (5.1), we have

$$\begin{aligned} &f(S - \frac{r}{n})g(X,Y) + \frac{\Delta f}{n}g(X,Y) \\ &+ S(X,Y) - \rho r g(X,Y) + \lambda g(X,Y) = 0, \end{aligned} \tag{5.2}$$

for all smooth vector fields X, Y on M. Now, combining equations (2.4) and (5.2), we get

$$S(X,Y) = \frac{1}{f+1}\left\{\frac{r-2\rho}{2(n-1)}f - \lambda + \rho r\right\} g(X,Y), \tag{5.3}$$

for all smooth vector fields X, Y on M. Hence, the Ricci tensor is proportional to the metric tensor, which shows that M is an Einstein manifold. This completes the proof. □

Remark 5.2. *(i) Contracting equation (5.3) over X and Y yields the soliton constant of gradient Ricci-Bourguignon soliton as*

$$\lambda = \frac{(r-2\rho)}{2(n-1)}f - \frac{r(f+1)}{n} + \rho r. \tag{5.4}$$

(ii) For a stiff matter fluid, we have $\sigma = \rho$. Moreover, for the spatial factor, $\sigma = \frac{r}{2}$, and hence the isotropic pressure $\rho = \frac{r}{2}$. Using $\rho = \frac{r}{2}$ in equation (5.4), we obtain

$$\lambda = -\frac{r(f+1)}{n} + \frac{r^2}{2}. \tag{5.5}$$

Thus, if the base manifold (M^n, g) of a static perfect fluid spacetime $(\tilde{M}^{n+1}, \tilde{g})$ admitting gradient Ricci-Bourguignon soliton corresponds to a stiff matter fluid, then the soliton is shrinking, provided that the scalar curvature is positive and f is large than r.

Definition 5.3. *[16] A perfect fluid spacetime is said to satisfy the timelike convergence condition if the Ricci tensor S satisfies $S(X,X) > 0$ for all timelike vector fields X on M.*

Theorem 5.4. *Let (M^n, g) be the base manifold of a static perfect fluid spacetime $(\tilde{M}^{n+1}, \tilde{g})$. If (M^n, g) admits a gradient Ricci-Bourguignon soliton, then it satisfies the timelike convergence condition if and only if*

$\lambda > \frac{2\rho-r}{2(n-1)}f - \rho r..$

Proof. Putting $X = Y = \xi$ in equation (5.3), we have

$$S(\xi,\xi) = \frac{1}{f+1}\left\{\frac{r-2\rho}{2(n-1)}f - \lambda + \rho r\right\} g(\xi,\xi). \tag{5.6}$$

Assume that the base manifold (M^n, g) of the static perfect fluid spacetime $(\tilde{M}^{n+1}, \tilde{g})$ satisfies the timelike convergence condition, that is, $S(\xi,\xi) > 0$. Then from equation (5.6), we have

$$\frac{1}{f+1}\left\{\lambda - \rho r + \frac{r-2\rho}{2(n-1)}f\right\} > 0,$$

which implies

$$\lambda > \frac{2\rho - r}{2(n-1)}f - \rho r.$$

Then equation (5.6) implies $S(\xi,\xi) > 0$, and hence the base manifold obeys the timelike convergence condition. This completes the proof. □

Proposition 5.5. *In [36] Yadav and Saxena shows that if* (M^n, g) *be the base manifold of a static perfect fluid spacetime* $(\tilde{M}^{n+1}, \tilde{g})$*. If* (M^n, g) *admits a unit torse-forming vector field* ξ*, then the curvature tensor* R *and the Ricci tensor* S *satisfy*

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{5.7}$$

and

$$S(X,\xi) = (n-1)\eta(X), \tag{5.8}$$

for all smooth vector fields X, Y *on* M.

Theorem 5.6. *If the base manifold* (M^n, g) *of a SPF spacetime* $(\tilde{M}^{n+1}, \tilde{g})$ *is admitting gradient Ricci-Bourguignon soliton and the basic vector field* ξ *is torseforming vector field then either the torseforming vector field is an orthogonal vector field or the soliton constant is*

$$\lambda = \frac{(2\rho - r)f + 2(n-1)\rho r + 2(n-1)^2(f+1)}{2(n-1)}.$$

Proof. Putting $Y = \xi$, in equation (5.3), we have

$$S(X,\xi) = \left\{\frac{r-2\rho}{2(n-1)}f - \lambda + \rho r\right\}\eta(X). \tag{5.9}$$

Combining equations (5.8) and (5.9), we have

$$\left[(n-1) - \left\{\frac{r-2\rho}{2(n-1)}f + \lambda - \rho r\right\}\right]\eta(X) = 0. \tag{5.10}$$

which provided the torseforming vector field is an orthogonal vector field or

$$\lambda = \frac{(2\rho - r)f + 2(n-1)\rho r + 2(n-1)^2(f+1)}{2(n-1)}. \tag{5.11}$$

□

Remark 5.7. *Since for stiff matter fluid* $\sigma = \rho$ *and for SPF spacetime* $\sigma = \frac{r}{2}$ *which implies isotropic pressure* $\rho = \frac{r}{2}$. *Then by equation (5.11), we have*

$$\lambda = \frac{r^2}{2} + (n-1)(f+1). \tag{5.12}$$

i.e., If the base manifold (M^n, g) *of SPF spacetime* $(\tilde{M}^{n+1}, \tilde{g})$ *is admitting gradient Ricci-Bourguignon soliton is a stiff matter fluid and the basic vector field is torseforming vector field then the soliton is always expanding.*

Theorem 5.8. *Let the base manifold* (M^n, g) *of a static perfect fluid spacetime* $\tilde{M}^{n+1} = M \times_f \mathcal{R}$ *admit a steady gradient Ricci-Bourguignon soliton. If the base manifold is Ricci symmetric with constant scalar curvature, then either the warping function* f *is constant or*

$$\rho r = \frac{r - 2\rho}{2(n-1)}.$$

Proof. Let the base manifold (M^n, g) of the static perfect fluid spacetime $\left(\tilde{M}^{n+1}, \tilde{g}\right)$ be Ricci symmetric. Then the Ricci tensor satisfies

$$(\nabla_X S)(Y, Z) = 0, \tag{5.13}$$

for all smooth vector fields X, Y, Z on M.

Taking the covariant derivative of equation (5.3), we have

$$(\nabla_X S)(Y, Z) = \left\{ \frac{r - 2\rho}{2(n-1)} - \lambda + \rho r \right\} \frac{(Xf)}{(f+1)^2} g(Y, Z), \tag{5.14}$$

for all smooth vector fields X, Y, Z on M.

Comparing equations (5.13) and (5.14), we get

$$\left\{ \frac{r - 2\rho}{2(n-1)} - \lambda + \rho r \right\} \frac{(Xf)}{(f+1)^2} = 0. \tag{5.15}$$

This implies either $Xf = 0$, that is, the warping function f is constant, or

$$\lambda = \rho r + \frac{r - 2\rho}{2(n-1)}. \tag{5.16}$$

Since the soliton is a steady gradient Ricci-Bourguignon, we have $\lambda = 0$. Hence equation (5.16) yields

$$\rho r = \frac{r - 2\rho}{2(n-1)}.$$

Hence the theorem. □

Theorem 5.9. *Let the base manifold* (M^n, g) *of a static perfect fluid spacetime* $\tilde{M}^{n+1} = M \times_f \mathcal{R}$ *admit a gradient Ricci-Bourguignon and let the basic vector field* ξ *be a torseforming vector field. If the base manifold is weakly Ricci symmetric with constant scalar curvature, then*

$$\xi(f) = \frac{-2(f+1)(n-1)\{r(n+1) + \rho\xi(r)\}}{2\lambda(n-1) - r(1-2\rho)f}. \tag{5.17}$$

Proof. Assume that the base manifold (M^n, g) of a static perfect fluid spacetime $\left(\tilde{M}^{n+1}, \tilde{g}\right)$ is weakly Ricci symmetric and admits a gradient Ricci-Bourguignon soliton. Then from equation (5.1), we have

$$\nabla_X Df + QX + (\lambda - \rho r)X = 0, \tag{5.18}$$

for all smooth vector fields X on M, where Q denotes the Ricci operator. Using equation (5.18) in the definition of the Riemannian curvature tensor, we obtain

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y. \tag{5.19}$$

Since the base manifold is weakly Ricci symmetric, equation (2.17) gives

$$QX = -r\,\eta(X)\xi. \tag{5.20}$$

Taking the covariant derivative of equation (5.20) along Y and using the torseforming property of ξ, we get

$$(\nabla_Y Q)X = -Y(r)\eta(X)\xi - r\left\{g(X,Y)\xi + \eta(X)Y + 2\eta(X)\eta(Y)\xi\right\}. \tag{5.21}$$

Interchanging X and Y in equation (5.23), we have

$$(\nabla_X Q)Y = -X(r)\eta(Y)\xi - r\left\{g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi\right\}, \tag{5.22}$$

Combining equations (5.19), (5.21) and (5.22), we have

$$R(X,Y)Df = r\left\{\eta(Y)X - \eta(X)Y\right\} + \rho\{Y(r)X - X(r)Y\}. \tag{5.23}$$

Contracting equation (5.23) with respect to X and using a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$, we get

$$S(Y,Df) = r\left\{n\,\eta(Y) + g(Y,e_i)\right\} + \rho\{Y(r)Df - g(Df,Y)\nabla r\}. \tag{5.24}$$

On the other hand, setting $X = Df$ in equation (5.3), we obtain

$$S(Y,Df) = \frac{1}{f+1}\left\{\frac{r-2\rho}{2(n-1)}f - \lambda + \rho r\right\}g(Y,Df). \tag{5.25}$$

Comparing equations (5.24) and (5.25) and putting $Y = \xi$, we obtain

$$\xi(f) = \frac{-2(f+1)(n-1)\{r(n+1)+\rho\xi(r)\}}{2\lambda(n-1)-r(1-2\rho)f}. \tag{5.26}$$

which yields equation (5.17). This completes the proof. □

Theorem 5.10. *Let the base manifold (M^n, g) of a static perfect fluid spacetime $\tilde{M}^{n+1} = M \times_f \mathcal{R}$ admit an gradient Ricci-Bourguignon soliton. If the base manifold is weakly Ricci symmetric, then the soliton constant λ is given by*

$$\lambda = \frac{r}{n}f - \frac{\Delta f}{n} + r(f+1) - \rho r f. \tag{5.27}$$

Proof. Using equations (2.3) and (5.1), we obtain

$$(\nabla^2 f)(X,Y) = \frac{1}{(f+1)}\left(\Delta f - rf - n\rho r f - n\lambda f\right) g(X,Y). \tag{5.28}$$

Since the base manifold (M^n, g) is weakly Ricci symmetric, equation (2.17) together with (5.1) yields

$$(\nabla^2 f)(X,Y) = r\eta(X)\eta(Y) - (\lambda + \rho r)g(X,Y). \tag{5.29}$$

Comparing equations (5.28) and (5.29) and setting $Y = \xi$, we obtain

$$\lambda = \frac{r}{n}f - \frac{\Delta f}{n} + r(f+1) - \rho r f.$$

This completes the proof. □

Lemma 5.11. *([5],[36]) Let $(V, \langle\cdot,\cdot\rangle)$ be an oriented four-dimensional inner product space of neutral signature. Then the following statements hold.*

1. *An algebraic curvature tensor R is self-dual if and only if, for any positively oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$, we have*

   $$K(e_1, e_i, x, y) = \omega_{ijk}\, \varepsilon_j \varepsilon_k\, K(e_j, e_k, x, y), \tag{5.30}$$

   for all $x, y \in V$, where $i, j, k \in \{2, 3, 4\}$, ω_{ijk} denotes the sign of the corresponding permutation and $\varepsilon_j = \langle e_j, e_j\rangle$.

2. *An algebraic curvature tensor R is self-dual if and only if, for a positively oriented pseudo-orthonormal basis $\{t, u, v, w\}$ (that is, $\langle t, v\rangle = \langle u, w\rangle = 1$ are the only non-zero inner products), the Conharmonic curvature tensor satisfies*

   $$K(t,v,x,y) = K(u,w,x,y), \qquad K(t,w,x,y) = 0, \qquad K(u,v,x,y) = 0, \tag{5.31}$$

   for all $x, y \in V$.

Theorem 5.12. *If a* 4*-dimensional half conharmonically flat base manifold* (M^n, g) *of a static perfect fluid spacetime* $\tilde{M}^{n+1} = M \times_f \mathcal{R}$, *admitting an gradient Ricci-Bourguignon soliton of neutral signature, is a stiff matter fluid, then the soliton is expanding, steady according as* $r > 0$, $r < 0$ *or* $r = 0$ *provided* $\|\nabla f\| = 0$.

Proof. Since $\|\nabla f\|^2 = g(\nabla f, \nabla f) = 0$, taking the covariant derivative provided

$$(\nabla^2 f)(\nabla f, X) = 0, \tag{5.32}$$

for all smooth vector fields X on M. Combining equation (5.34) with equation (5.1) and using the definition of the Ricci operator, we obtain

$$Q(\nabla f) = (\lambda - \rho r)\nabla f. \tag{5.33}$$

Since the base manifold (M^n, g) is 4-dimensional and half conharmonically flat, Suppose that it is self-dual. Let $\{\nabla f, e_1, e_2, e_3\}$ be a pseudo-orthonormal frame. Then, from equations (5.35) and (5.1), we have

$$\begin{aligned} K(\nabla f, e_2, X, Y) = 0, \quad K(e_1, e_3, X, Y) &= 0, \\ K(\nabla f, e_3, X, Y) = 0, \quad K(e_1, e_2, X, Y) &= 0. \end{aligned} \tag{5.34}$$

Taking $Y = \nabla f$ in the first equation of (5.34) and using equations (2.22) and (5.33), we obtain

$$K(\nabla f, e_2, X, \nabla f) = \left(\lambda - \rho r + \frac{\Delta f}{2f}\right) g(\nabla f, X). \tag{5.35}$$

for all smooth vector fields X on M.

Next, putting $Y = \nabla f$ in the second equation of (5.34) and using equations (2.22) and (5.33), we get

$$K(e_1, e_3, X, \nabla f) = 0. \tag{5.36}$$

Combining equations (5.35) and (5.36), we obtain either

$$\lambda = \rho r - \frac{1}{2} + \left(\frac{r}{6} - \frac{4\rho}{3}\right),$$

or $g(\nabla f, X) = 0$ for all X, which contradicts $\nabla f \neq 0$. Hence,

$$\lambda = \rho r - \frac{1}{2} + \left(\frac{r}{6} - \frac{4\rho}{3}\right). \tag{5.37}$$

Since the base manifold is a stiff matter fluid, we have $\rho = \frac{r}{2}$, substituting this in (5.37), we get

$$\lambda = \frac{r(2r+1)}{4}. \tag{5.38}$$

Thus, the soliton is expanding, steady according as $r > 0$, $r < 0$ or $r = 0$. This completes the proof. □

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