

A Numerical Approximation for Burgers Equation

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Abstract

This paper gives an approximation of Burgers Equation based on Time Discretization and Variational Method. The approximations so obtained are compared with the exact solutions and it is found that they are in good agreement. Also graphs for various values of Viscosity and Time are drawn. This Paper also demonstrates the advantage of availability of Mathematical Software like Mathematica.

Keywords: Burgers Equation, Time Discretization, Variational Method.

AMS Subject Classification: 35Q35, 65N20

Introduction

Burgers equation first appeared in a paper by Bateman [8]. Later on J.M. Burger [9] investigated various aspects of turbulence and used this equation to model turbulence. Cole [1] studied the properties exhibited by Burgers equation. He gave the exact solution to the following problem of Burgers equation with the associated Boundary and Initial conditions:

$$u_t + uu_x = \varepsilon u_{xx}, 0 < x < 1, t > 0 \quad (1)$$

$$u(x, 0) = \sin \pi x, 0 < x < 1 \quad (2)$$

$$u(0, t) = u(1, t) = 0, t > 0 \quad (3)$$

The exact solution to this problem (1),(2) and (3) given by him can be stated as:

$$u(x, t) = 2\pi\varepsilon \frac{\sum_{n=1}^{\infty} a_n e^{-n^2\pi^2\varepsilon t} n \sin n\pi x}{a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2\varepsilon t} \cos n\pi x} \quad (4)$$

where

$$a_0 = \int_0^1 e^{-\frac{(1-\cos \pi x)}{2\pi\varepsilon}} dx \quad (5)$$

and

$$a_n = 2 \int_0^1 e^{-\frac{(1-\cos \pi x)}{2\pi\varepsilon}} \cos n\pi x dx, n \geq 1 \quad (6)$$

But it is a fact that the exact solution to Burgers equation is limited to certain values of viscosity ε . For small values of ε , numerical methods are more appropriate and so, many numerical methods such as Variational Iteration Method [3], Meshless Method [5], Galerkin Finite Element Approach [4] etc have been developed recently. This Paper is also another attempt to solve Burgers equation numerically.

Description of the method

The Method of Discretization in Time converts a Partial Differential Equation to a set of Ordinary Differential Equations. The steps involved in it can be summarized as follows:

Firstly the time interval is divided into p subintervals of length Δt . Secondly u_t is changed to $[z_j(x) - z_{j-1}(x)] / \Delta t$, where z_j is the approximate solution at the point of division $t_j = j\Delta t, (j=1, 2, \dots, p)$, thus converting the Partial Differential Equation into a set of p - Ordinary Differential Equations. Lastly we seek the solution at each time step starting from the initial condition $z(0) = u(x, 0)$.

Following the First and Second steps the Burgers equation (1) along with the initial and boundary conditions (2) and (3) reduces to:

$$-\varepsilon z_j'' + z_j z_j' + \frac{1}{\Delta t} z_j = \frac{1}{\Delta t} z_{j-1} \quad (7)$$

$$z_0(x) = \sin \pi x \quad (8)$$

$$z_j(0) = z_j(1) = 0 \quad (9)$$

This boundary problem could be solved exactly or numerically. But the solution becomes more difficult with increasing j when we solve exactly. So, we apply a Variational method (Galerkin) constructed on time discretization.

We choose the weighted function,

$$w = \psi_i = \cos(1 - 2ix) \frac{\pi}{2}, i = 1, 2, \dots, m \quad (10)$$

for the variational problem

$$\int_0^1 w(-\varepsilon z_j'' + z_j z_j' + \frac{1}{\Delta t} z_j - \frac{1}{\Delta t} z_{j-1}) dx = 0, j = 1, 2, \dots, p \quad (11)$$

where the Galerkin approximation z_j^m of the function $z_j(x)$ in (11) is

$$z_j^m(x) = \sum_{i=1}^m c_i^j \psi_i \tag{12}$$

Integrating (11) we get,

$$\int_0^1 (\varepsilon w' z_j' + w z_j z_j' + \frac{1}{\Delta t} w z_j - \frac{1}{\Delta t} w z_{j-1}) dx = 0 \tag{13}$$

where we have used the fact that $w(0) = w(1) = 0$. After substituting (10) and (12) in (13) and using the following results:

$$\int_0^1 \cos[(1-2ix)\frac{\pi}{2}] \cos[(1-2nx)\frac{\pi}{2}] dx = \begin{cases} 1/2, & i = n \\ 0, & i \neq n \end{cases} \tag{14}$$

$$\int_0^1 \sin[(1-2ix)\frac{\pi}{2}] \sin[(1-2nx)\frac{\pi}{2}] dx = \begin{cases} 1/2, & i = n \\ 0, & i \neq n \end{cases} \tag{15}$$

$$\int_0^1 \cos[(1-2ix)\frac{\pi}{2}] \cos[(1-2nx)\frac{\pi}{2}] \sin[(1-2kx)\frac{\pi}{2}] dx = \begin{cases} -1/4, & i = k - n \\ 1/4, & i = n - k \\ 1/4, & i = n + k \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

we see that for each i of w , equation (13) is a nonlinear equation and for each j the corresponding set of equations of (13) gives a set of nonlinear equations in c_i . Solving this set of nonlinear equations in Mathematica Computer Software using Newton's Method we get the approximate solution for the various values of m , ε , Δt , and p . The solutions obtained for $\varepsilon = 1$ and $\varepsilon = 0.1$ are compared with the exact solution (Table 1) and it is found that they are in good agreement.

Table 1: Comparison of Numerical solution and Exact solution.

x	t	$\varepsilon = 1$		$\varepsilon = 0.1$	
		Numerical Solution	Exact Solution	Numerical Solution	Exact Solution
0.25	0.4	0.3091	0.3088	0.0138	0.0135
	0.6	0.2409	0.2407	0.0019	0.0018
	0.8	0.1958	0.1956	0.0002	0.0002
	1.0	0.1627	0.1625	0.0000	0.0000
0.5	0.4	0.5699	0.5696	0.0196	0.0192
	0.6	0.4475	0.4472	0.0027	0.0026
	0.8	0.3595	0.3592	0.0003	0.0003
	1.0	0.2921	0.2919	0.0000	0.0000
0.75	0.4	0.6253	0.6254	0.0138	0.0136
	0.6	0.4873	0.4872	0.0019	0.0018
	0.8	0.3741	0.3739	0.0002	0.0002
	1.0	0.2877	0.2874	0.0000	0.0000

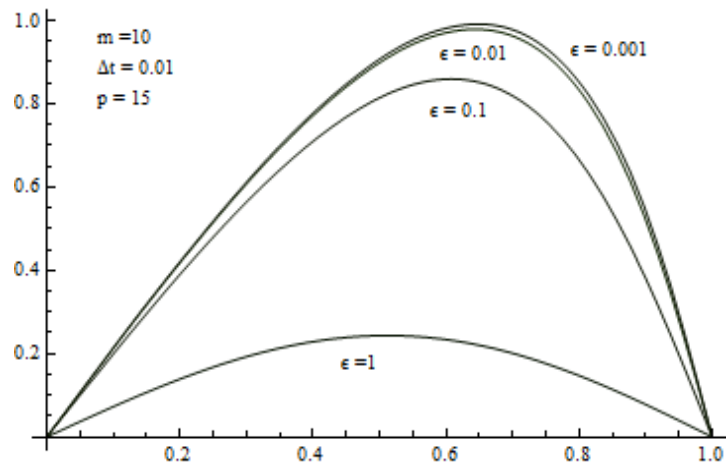


Figure 1: Graph for various values of ε .

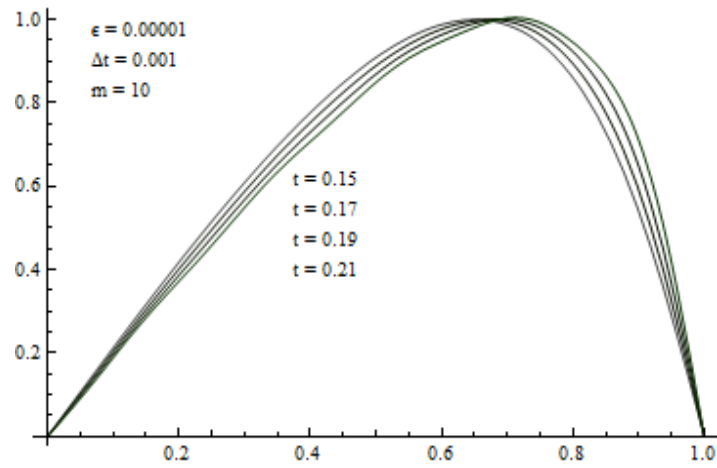


Figure 2: Graph for different times for a given small ε .

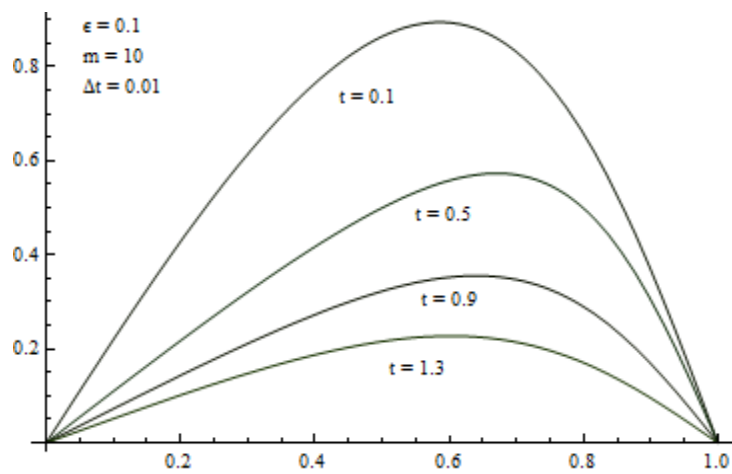


Figure 3: Graph for larger time step and larger ε .

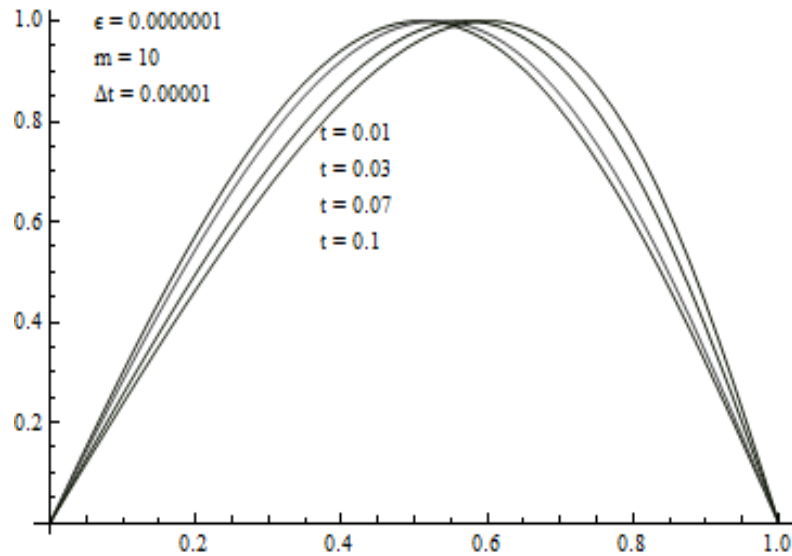


Figure 4: Graph for small time step and small ϵ .

Conclusion

We see that Finite Element Galerkin Method constructed on Time Descretization effectively approximates the solution of Burgers equation (1), (2) and (3) numerically as shown in Table [1]. The solution obtained is more accurate than the one given in [2]. The effectiveness of the method is supported by the graphs (Fig. 1, 2, 3, 4) shown for various values of ϵ and t . All computational works were carried out in Mathematica7, which shows the advantage of availability of such mathematical software in performing large computational works.

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