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A Note on Tensor Product of G-Frames

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Abstract

The tensor product of g-frames in tensor product of Hilbert spaces is introduced. It was shown that the tensor product of two g-frames is a g-frame for the tensor product of Hilbert spaces. The concept of tensor product of g-frame operator on tensor product of Hilbert space is given and results of it are presented.

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Introduction

Peter G. Casazza [1] presented a tutorial on frame theory and he suggested the major directions of research in frame theory. A. Najati and A. Rahimi [2] have developed the generalized frame theory and introduced methods for generating g-frames of a Hilbert space. G-frames are generalization of frames. W.Sun [3] presented characterizations of g-frames and proved that g-frames share many useful properties with frames. The tensor product of frames in tensor product of Hilbert spaces is introduced by G.Upender Reddy and N.Gopal Reddy [4]. They proved that the tensor product of two frames is a frame for the tensor product of Hilbert spaces and results on tensor frame operator are presented.

The tensor product of g-frames in tensor product of Hilbert spaces is introduced. It was shown that the tensor product of two g-frames is a g-frame for the tensor product of Hilbert spaces. The concept of tensor product of g-frame operator on tensor product of Hilbert space is given and results of it are presented.

Preliminaries

Thought this section $\{H_j, j \in J\}$ will denote a sequence of Hilbert spaces.

Let $L(H,H_j)$ be a collection all bounded linear operators from H to H_j and $\{\Lambda_j \in L(H,H_j): j \in J\}$.

Definition 2.1. A sequence of operators $\{\Lambda_j\}_{j\in J}$ is said to be g-frame for Hilbert space H with respect to sequence of Hilbert spaces $\{H_j, j \in J\}$, if there exist two constants $0 < A \le B < \infty$, for any vector $h \in H$,

$$A\,\|h\|^2 \le \sum_{j\in J} \|\Lambda_j h\|^2 \le B\,\|h\|^2 .$$

The above inequality is called a g-frame inequality. The numbers A and B are called the lower frame bound and upper frame bound respectively

Definition 2.2. A g-frame $\{\Lambda_j\}_{j\in J}$ for H is said to be g-tight frame if A = B. Then we have

$$A\,\|h\|^2 = \sum_{j\in J} \|\Lambda_j h\|^2 \text{, for all } h \in H.$$

Definition 2.3. A g-frame $\{\Lambda_j\}_{j\in J}$ for H is said to be g-normalized tight frame for H if A = B = 1. Then we have,

$$\|h\|^2 = \sum_{j\in J} \|\Lambda_j h\|^2 \text{, for all } h \in H.$$

Definition 2.4. Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for Hilbert space H. A g-frame operator

$S^g : H \to H$ is defined as

$$S^g h = \sum_{j\in J} \Lambda_j^* \Lambda_j h \text{, for all } h \in H.$$

By using above definitions the following theorem on g-frame operator can be derived easily, so left to reader.

Theorem 2.5. Suppose $\{\Lambda_j\}_{j\in J}$ is a g-frame iff $A\, I_{op} \le S^g \le B\, I_{op}$.

Tensor product of g-frames

In this section the tensor product of g-frames in tensor product of Hilbert spaces is introduced. It was shown that the tensor product of two g-frames is a g-frame for the tensor product of Hilbert spaces. The concept of tensor product of g-frame operator on tensor product of Hilbert space is given and results of it are presented.

Let H and K be two Hilbert spaces with inner products $<.,.>_1$, $<.,.>_2$ and norms $\|.\|_1, \|.\|_2$ respectively. The tensor product of H and K is denoted by $H \otimes K$ and is an inner product space with respect to the inner product

$$< h_1 \otimes h_2 , k_1 \otimes k_2 > = <h_1, k_1 >_1 < h_2 , k_2 >_2$$

for all $h_1, k_1 \in H_1$ and $h_2, k_2 \in H_2$. The norm on $H \otimes K$ is defined by

$$\|h \otimes k\| = \|h\|_1 \; \|k\|_2 \quad \forall \; h \in H, k \in K.$$

The space $H \otimes K$ is clearly completion with the above inner product. Therefore the space $H \otimes K$ is a Hilbert space. We denote L (H, K) be the space of all bounded linear operators from $H \to K$. Let $M \in L$ (H) and $N \in L$ (K) be two operators, then the tensor product of operator $M \otimes N$ acts on $H \otimes K$ as

$$(M \otimes N)(h \otimes k) = Mh \otimes Nk$$

for every $h \in H$, $k \in K$ and $h \otimes k \in H \otimes K$.

We note that if $M_1, M_2 \in L$ (H), $N_1, N_2 \in L(K)$ and $M_1 \otimes N_1$, $M_2 \otimes N_2 \in L(H \otimes K)$ then $(M_1 \otimes N_1)(M_2 \otimes N_2) = M_1 M_2 \otimes N_1 N_2$.

In this paper we denote I_H is the identity operator on H and I_K is the identity operator on K then $I_H \otimes I_K = I_{H \otimes K}$ is the identity operator on $H \otimes K$.

The following is the extension of (2.1) to the sequence of operators $\{\Lambda_i \otimes ß_j\}$.

Definition 3.1. Let $\{\Lambda_i\}$ and $\{ß_j\}$ be the sequences of operators in Hilbert spaces H and K respectively. Then the sequence of operators $\{\Lambda_i \otimes ß_j\}$ is said to be a tensor product of g-frame for the tensor product of Hilbert spaces $H \otimes K$, if there exist two constants $0 < A \le B < \infty$, such that

$$A \|h \otimes k\|^2 \le \sum_{i,j} \|(\Lambda_i \otimes ß_j)(h \otimes k)\|^2 \le B \|h \otimes k\|^2, \text{ for all } h \otimes k \in H \otimes K.$$

The numbers A and B are called lower and upper frame bounds of the tensor product of g-frame respectively.

Theorem 3.2. Let $\{\Lambda_i\}$ and $\{ß_j\}$ be two g-frames for Hilbert spaces H and K with respect to $\{H_i\}$ and $\{K_j\}$, respectively. Then $\{\Lambda_i \otimes ß_j\}$ is a tensor product of g-frame for $H \otimes K$ with respect to $\{H_i \oplus K_j\}$.

Proof. Let $\{\Lambda_i\}$ be a g-frame for H with frame bounds A_1 and B_1 with respect to $\{H_i\}$ then, for all $h \in H$

$$A_1 \|h\|^2 \leq \sum_i \|\Lambda_i h\|^2 \leq B_1 \|h\|^2 \tag{3.3}$$

Let $\{ ß_j \}$ be a g-frame for K with frame bounds A_2 and B_2 with respect to $\{K_j\}$, then, for all $k \in K$

$$A_2 \|k\|^2 \leq \sum_j \|ß_j k\|^2 \leq B_2 \|k\|^2 \tag{3.4}$$

multiplying the equations (3.3) and (3.4), we get

$$A_1 A_2 \|h\|^2 \|k\|^2 \leq (\sum_i \|\Lambda_i h\|^2)(\sum_j \|ß_j k\|^2) \leq B_1 B_2 \|h\|^2 \|k\|^2$$

$$\Rightarrow A_1 A_2 \|h \otimes k\|^2 \leq \sum_{i,j} (\|\Lambda_i h\|^2 \|ß_j k\|^2) \leq B_1 B_2 \|h \otimes k\|^2, \text{ for all } h \otimes k \in H \otimes K$$

$$\Rightarrow A_1 A_2 \|h \otimes k\|^2 \leq \sum_{i,j} \|(\Lambda_i h \otimes ß_j k)\|^2 \leq B_1 B_2 \|h \otimes k\|^2, \text{ for all } h \otimes k \in H \otimes K$$

$$\Rightarrow A_1 A_2 \|h \otimes k\|^2 \leq \sum_{i,j} \|(\Lambda_i \otimes ß_j)(h \otimes k)\|^2 \leq B_1 B_2 \|h \otimes k\|^2,$$

for all $h \otimes k \in H \otimes K$

$\Rightarrow \{\Lambda_i \otimes ß_j\}$ is a tensor product of g-frame for $H \otimes K$. □

Theorem 3.5. If $\{\Lambda_i \otimes ß_j\}$ is a tensor product of g-frame for $H \otimes K$ with respect to $\{H_i \oplus K_j\}$. Then $\{\Lambda_i\}$ and $\{ß_j\}$ are g-frames for Hilbert spaces H and K with respect to $\{H_i\}$ and $\{K_j\}$, respectively.

Proof. Suppose that $\{\Lambda_i \otimes ß_j\}$ is a tensor product of g-frame for $H \otimes K$ with frame bounds A and B. Then for each $h \otimes k \in H \otimes K - \{0 \otimes 0\}$ for all $h \in H$, $k \in K$

$$A\|h \otimes k\|^2 \leq \sum_{i,j} \|(\Lambda_i \otimes ß_j)(h \otimes k)\|^2 \leq B\|h \otimes k\|^2, \text{ for all } h \otimes k \in H \otimes K.$$

$$\Rightarrow A\|h\|^2 \|k\|^2 \leq \sum_{i,j} \|(\Lambda_i h \otimes ß_j k)\|^2 \leq B\|h\|^2 \|k\|^2, \text{ for all } h \otimes k \in H \otimes K.$$

$$\Rightarrow A\|h\|^2 \|k\|^2 \leq (\sum_i \|\Lambda_i h\|^2)(\sum_j \|ß_j k\|^2) \leq B\|h\|^2 \|k\|^2, \text{ for all } h \otimes k \in H \otimes K.$$

Consider $h \otimes k$ is a non zero vector i.e. h and k are non zero vectors, therefore the above inequality becomes

$$\Rightarrow \frac{A\|k\|^2}{\sum_j \|ß_j k\|^2} \|h\|^2 \leq (\sum_i \|\Lambda_i h\|^2) \leq \frac{B\|k\|^2}{\sum_j \|ß_j k\|^2} \|h\|^2$$

$\Rightarrow A_1 \|h\|^2 \le \sum_i \|\Lambda_i h\|^2 \le B_1 \|h\|^2$, for all $h \in H$

where $A_1 = \dfrac{A\|k\|^2}{\sum_j \|ß_j k\|^2}$ and $B_1 = \dfrac{B\|k\|^2}{\sum_j \|ß_j k\|^2}$

which shows that $\{\Lambda_i\}$ is a g-frame for H. Similarly we can prove that $\{ß_j\}$ is a g-frame for K. □

Hence we can have the following remark.

Remark 3.6. If the sequence of operators $\{\Lambda_i\}$,$\{ß_j\}$ and $\{\Lambda_i \otimes ß_j\}$ are the frames for the Hilbert spaces H , K and $H\otimes K$ respectively and S^g_Λ, $S^g_ß$ and $S^g_{\Lambda\otimes ß}$ are the g-frame operators respectively of above frames, then from 2.4, we have the following.

$S^g_\Lambda h = \sum_i \Lambda_i^*\Lambda_i h$, $S^g_ß k = \sum_j ß_j^* ß_j k$ and

$S^g_{\Lambda\otimes ß}(h\otimes k) = \sum_{i,j} (\Lambda_i \otimes ß_j)^* (\Lambda_i \otimes ß_j)(h \otimes k)$

for all $h\in H$, $k\in K$ and $h\otimes k \in H\otimes K$.

Theorem 3.7. If $\{\Lambda_i\}$,$\{ß_j\}$ and $\{\Lambda_i \otimes ß_j\}$ are the g-frames for the Hilbert spaces H, K and $H\otimes K$ with g-frame operators S^g_Λ, $S^g_ß$ and $S^g_{\Lambda\otimes ß}$ respectively, then

$S^g_{\Lambda\otimes ß} = S^g_\Lambda \otimes S^g_ß$.

Proof. For $h\otimes k \in H\otimes K$, we have

$S^g_{\Lambda\otimes ß}(h\otimes k) = \sum_{i,j} (\Lambda_i \otimes ß_j)^* (\Lambda_i \otimes ß_j)(h \otimes k)$

$= \sum_{i,j} (\Lambda_i^* \otimes ß_j^*)(\Lambda_i h \otimes ß_j k)$

$= \sum_{i,j} (\Lambda_i^*\Lambda_i h \oplus ß_j^* ß_j k)$

$= \sum_i \Lambda_i^*\Lambda_i h \otimes \sum_j ß_j^* ß_j k$

$\Rightarrow S^g_{\Lambda\otimes ß} = S^g_\Lambda \otimes S^g_ß$. □

Theorem 3.8. If S^g_Λ and $S^g_ß$ are the g- frame operators for the g-frames $\{\Lambda_i\}$ and $\{ß_j\}$ respectively, then $A_1 A_2 I_{H\otimes K} \le S^g_{\Lambda\otimes ß} \le B_1 B_2 I_{H\otimes K}$.

Where I_H , I_k and $I_{H\otimes K}$ are the identity operators on H , K and $H\otimes K$ respectively.

Proof. Suppose $\{\Lambda_i\}$ is a g-frame for H with frame bounds A_1 and B_1 with respect to $\{H_i\}$ and $\{ß_j\}$ be a g-frame for K with frame bounds A_2 and B_2 with respect to $\{K_j\}$. Then by theorem 2.5, we have

$A_1 I_H \leq S^g_\Lambda \leq B_1 I_H$ and

$A_2 I_k \leq S^g_ß \leq B_2 I_k$

By taking the tensor product of above two inequalities, we obtain the required result. □

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