

A Study of Testing –Function Space $L_{+,a,\lambda}$.

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Abstract

Malgonde and Chaudhary [2] described a simple generalization of the two-sided Laplace Transformation defined by ,

$$G(s) = L_{\lambda}(g)(s) = \int_{-\infty}^{\infty} (st)^{\lambda} e^{-st} g(t) dt$$

For suitably restricted functions $g(t)$ where λ is any real number , is extended to certain class of generalized functions space by establishing an isomorphism between the space and the space of two-sided Laplace Transformable generalized function by Zemanian [5]

In this proposed work we shall study **Testing –Function Space $L_{+,a,\lambda}$** of certain class of distribution.

Introduction

Some generalization of the conventional Laplace transformation are given by many authors from time to time.

A simple generalization of the conventional two-sided Laplace transform

$$F(s) = L(f)(s) = \int_{-\infty}^{\infty} e^{-st} f(t) dt \quad (1.1)$$

$$\text{Is given by } G(s) = L_{\lambda}(g)(s) = \int_{-\infty}^{\infty} (st)^{\lambda} e^{-st} g(t) dt \quad (1.2)$$

Where $g(t)$ is suitably restricted conventional function on the real line $-\infty < t < \infty$ and λ is a real number.

If $G(s)$ is given by (1.2) then its complex inversion formula is

$$\frac{1}{2}[d(t+) + g(t-)] = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} (st)^{-\lambda} G(s) e^{st} ds \quad (1.3)$$

Where $s = \sigma + iT$

The transform (1.1) is already been studied in the sense of the generalized functions by Zemanian [4]. The transform (1.2) and its complex –inversion formula (1.3) and some operational transform formulae for (1.2) in the sense of generalized functions for a certain class , has been studied by Singh[3,4]

In the present paper , we develop an alternative approach to the one sided Laplace transformation (1.2). We define a certain countable –union space $L_{+\lambda}(w,z)$ that arise from the $L_{a,b,\lambda}$ spaces defined by Singh [3] to take our subsequent discussion on a simpler form. Moreover $L'_{\lambda}(w,z)$ spaces have nicer structure than do the $L'_{a,b,\lambda}$. my procedure for extending (1.2) is based on the idea of constructing a testing function space $L_{\lambda}(\sigma_1, \sigma_2)$ which is countable union space and contain the kernel $(st)^{\lambda} e^{-st}$ for $\sigma_1 < \text{Re } s < \sigma_2$. Then the generalized Laplace transform $G(s)$ of any member of the dual space $L'_{\lambda}(\sigma_1, \sigma_2)$ is obtained by applying $g(t)$ to the kernel $(st)^{\lambda} e^{-st}$

$$G(s) = \langle g(t) , (st)^{\lambda} e^{-st} \rangle , \quad \sigma_1 < \text{Re } s < \sigma_2 \quad (1.4)$$

It turns out that this transformation can be related to the two – sided Laplace transformation studied by Zemanian[5] and as a result , several properties of the transformation (1.4) can be carried over to the present case.

Let us consider a transformation

$$L_{+, \lambda}[f(t)] = F(s) = \int_0^{\infty} e^{-st} (st)^{\lambda} f(t) dt.$$

Where $f(t)$ is a suitably restricted conventional functional on the real line $0 < t < \infty$ and $f(t) = o(e^{at})$ as $t \rightarrow \infty$

1. Testing function space $L_{+,a,\lambda}$.

$L_{+, a, \lambda}$ denotes the space of all complex valued smooth functions

$\phi(t)$ on $0 < t < \infty$ on which the seminorm γ_k is defined as

$$\gamma_k(\phi) = \gamma_{a,+, \lambda, k}(\phi) = \sup_{0 < t < \infty} |e^{at} t^{k-\lambda} D^k \phi(t)| \dots\dots [1]$$

Thus ϕ and all its derivatives are of the order $o(e^{-at})$ as $t \rightarrow \infty$

$L_{+, a, \lambda}$ is a linear space

Moreover, each $\gamma_{+, a, k, \lambda}$ is seminorm on $L_{+, a, \lambda}$ and the collection of all

such seminorms separates $L_{+, a, \lambda}$ since the assumption that $\gamma_{+, a, 0, \lambda}(\phi) = 0$

implies that $\phi(t) = 0$ on $0 < t < \infty$.

We assign to $L_{+, a, \lambda}$ the topology generated by the multinorm

$\{\gamma_{+, a, k, \lambda}\}_{k=0}^{\infty}$. Therefore $L_{+, a, \lambda}$ is a countable multinormed space.

Lemma:1) $L_{+,a,\lambda}$ is complete.

Proof : We define $\gamma_{n,\lambda} = \gamma_{a,n,\lambda}(\phi) = \sup_{0 < t < \infty} |e^{at} t^{k-\lambda} \phi(t)|$

The sequence $\{\phi_n\}$ is called cauchy sequence in $L_{+,a,\lambda}$ if and only if each $\phi_n \in L_{+,a,\lambda}$ and for each fixed k the function $e^{at} t^{k-\lambda} D^k \phi_n(t)$ comprise a uniform cauchy seqense on $0 < t < \infty$ as $n \rightarrow \infty$.

In this case it follows from standered theroem of calculus that there exist a smoot function $\phi(t)$ such that ,for each k and t,

$D^k \phi_n(t) \rightarrow D^k \phi(t)$ as $t \rightarrow \infty$. Moreover given $\varepsilon > 0$ there exist an N_k such that ,for every $n, m > N_k$

$$\left| e^{at} t^{k-\lambda} D^k (\phi_n(t) - \phi_m(t)) \right| \leq \varepsilon, 0 < t < \infty$$

$$m, n > N_k \dots\dots\dots(1)$$

Taking the limit as $m \rightarrow \infty$ we obtain

$$\left| e^{at} t^{k-\lambda} D^k (\phi_n(t) - \phi(t)) \right| \leq \varepsilon, 0 < t < \infty \text{ and } n < N_k$$

Thus as $n \rightarrow \infty$ $\gamma_k(\phi_n - \phi) \rightarrow 0$ for each k(2)

Finally because of the uniformity of the convergence and the fact that each $e^{at} t^{k-\lambda} D^k \phi_n(t)$ is bounded on $0 < t < \infty$, there exist a constant c_k not depending on n such that , $|e^{at} t^{k-\lambda} D^k \phi_n(t)| < c_k$.for all t.

Therefore eqetion(1) implies that $|e^{at} t^{k-\lambda} \phi(t)| \leq c_k + \varepsilon$, which shows that the limit function ϕ is member of $L_{+,a,\lambda}$. Hence $L_{+,a,\lambda}$ is complete. and therefore a Frechet space.

$L'_{+,a,\lambda}$ denote the dual space of $L_{+,a,\lambda}$.

Since $L_{+,a,\lambda}$ is complete $L'_{+,a,\lambda}$ is also complete. $L'_{+,a,\lambda}$ is a generalized function space.If $a \leq c$ then $L_{+,c,\lambda} \subset L_{+,a,\lambda}$, and the topology of $L_{+,c,\lambda}$ is stronger than the topology induced on $L_{+,c,\lambda}$ by $L_{+,a,\lambda}$.

To see this note that $0 < e^{at} < e^{ct}$ on $0 < t < \infty$.

$$\therefore |e^{at} t^{k-\lambda} D^k \phi(t)| \leq |e^{ct} t^{k-\lambda} D^k \phi(t)|$$

so that $\gamma_{+,a,k}(\phi) \leq \gamma_{+,c,k}(\phi)$.

Let $\phi \in L_{+,a,b,\lambda}$ where b is arbitrary then,

$$\rho_{+,a,k}(\phi) = \sup_{0 < t < \infty} \left| \frac{e^{at}}{K_{a,b}(t)} K_{a,b}(t) t^{k-\lambda} D^k \phi(t) \right|$$

$$\leq C \gamma_{a,b,k}(\phi), \text{ where } c = \sup_{0 < t < \infty} \left| \frac{e^{at}}{K_{a,b}(t)} \right|$$

$\therefore L_{+,a,b,\lambda} \subset L_{+,a,\lambda}$, and the topology of $L_{+,a,b,\lambda}$ is stronger than that induced on it by $L_{+,a,\lambda}$.

It follows that the restriction of any $f \in L'_{+,a,\lambda}$ to $L_{+,a,b,\lambda}$ is a member of $L'_{+,a,b,\lambda}$, no matter what the choice of b is.

Lemma:2) $t^\lambda e^{-st} \in L_{+,a,\lambda}$ for all real λ .

Proof :-To prove $t^\lambda e^{-st} \in L_{+,a,\lambda}$ it is sufficint to prove that

$$\gamma_n(t^\lambda e^{-st}) = \sup_{0 < t < \infty} |e^{at} t^{n-\lambda} D^n (t^\lambda e^{-st})|$$

is bounded for all real λ .

$$\text{But } D^n (t^\lambda e^{-st}) = \sum_{k=0}^n c_k D^{n-k} t^\lambda D^k e^{-st},$$

where $n=0, 2, 3, \dots$ $k=0, 1, 2, 3, \dots$

$$= \sum_{k=0}^n c_k \lambda(\lambda-1)(\lambda-2)(\lambda-3)\dots[\lambda-(n-k-1)] t^{\lambda-n+k} e^{-st} (-s)^k.$$

$$= \sum_{k=0}^n c_{n,k} t^{\lambda-n+k} e^{-st} (-s)^k.$$

where $c_{n,k}$ are constants depending upon n & k.

$$t^{n-\lambda} D^n (t^\lambda e^{-st}) = \sum_{k=0}^n c_{n,k} t^{n-\lambda+\lambda-n+k} e^{-st} (-s)^k.$$

$$= \sum_{k=0}^n c_{n,k} t^k e^{-st} (-s)^k.$$

$$e^{at} t^{n-\lambda} D^n (t^\lambda e^{-st}) = \sum_{k=0}^n e^{at} c_{n,k} t^k e^{-st} (-s)^k.$$

$$\therefore |e^{at} t^{n-\lambda} D^n (t^\lambda e^{-st})| = M \sum_{k=0}^n |t^k e^{-(s-a)t}|$$

But each of $|t^k e^{-(s-a)t}|$ is bounded

for all $k=0, 1, 2, \dots$ & $\text{Re}.s > a$.

$\therefore \sup_{0 < t < \infty} |e^{at} t^{n-\lambda} D^n (t^\lambda e^{-st})|$ is bounded

for all real values λ & $\text{Re}.s > a$

Hence the result .

Let $\{a_n\}_{n=1}^\infty$ be a monotonic sequence of real numbers such that $a_n \rightarrow w^+$ where n denotes a finite number or $-\infty$.

Then we define $L_{+, \lambda}(w) = \bigcup_{n=1}^\infty L_{+, a_n, \lambda}$ is called

countable union space of all $L_{+, a_n, \lambda}$ spaces.

A sequence converges in $L_{+, \lambda}(w)$ if and

only if it converges in one of the $L_{+, a_n, \lambda}$ spaces.

For each $k=0,1,2,3,\dots$ $t^k e^{-st}$ is a member

$L_{+, \lambda}(w)$ if $\text{Re } s > w$.

Again $L'_{+, \lambda}(w)$ denotes dual of $L_{+, \lambda}(w)$.

Both $L_{+, \lambda}(w)$ and $L'_{+, \lambda}(w)$ are complete spaces.

If $w < u$ then $L_{+, \lambda}(u) \subset L_{+, \lambda}(w)$ and converges

in $L_{+, \lambda}(u)$ implies converges in $L_{+, \lambda}(w)$.

Thus restriction of any $f \in L'_{+, \lambda}(w)$ to $L_{+, \lambda}(u)$

is in $L_{+, \lambda}(u)$

Theorem: D is dense in $L_{+, \lambda}(w)$ for every w .

Proof :- Let ϕ be any element of $L_{+, \lambda}(w)$ then

$\phi \in L_{+, c, \lambda}$ for some $c > w$.

Choose a such that $w < a < c$, we prove our assertion

by constructing a sequence whose members are in D

and which converges to ϕ in $L_{+, a, \lambda}$, and therefore in $L_{+, \lambda}(w)$

Let $\lambda(t) \in D$ be such that $\lambda(t)=1$ for $|t| < 1$

and $\lambda(t)=0$ for $|t| > 2$.

Also let $n > 1$ we may write,

$$e^{at} t^{k-\lambda} D^k [\lambda(t/n)\phi(t) - \phi(t)]$$

$$= \sum_{m=1}^k c_m \{ D^{k-m} [\lambda(t/n) - 1] t^{k-\lambda} e^{at} D^m \phi(t) \}. \quad \text{--- (1)}$$

consequently as $n \rightarrow \infty$ the RHS of (1) converges uniformly

zero on $0 < t < \infty$.

Thus $\{\lambda(t/n)\phi(t)\}_{n=1}^\infty$ is the sequence which we seek.

* $L'_{+, \lambda}(w)$ is a subspace of D' .

Since D dense in $L_{+, \lambda}(w)$ the sequence $\{\phi_n\}$ in D

converges to ϕ in $L_{+, \lambda}(w)$.

Therefore the values that $f \in L'_{+, \lambda}(w)$ assigns to D

are uniquely determined the values that f assigns to $L_{+, \lambda}(w)$.

i.e. for $\phi_n \in D$ $\phi_n \rightarrow \phi$

in $L_{+, \lambda}(w)$ as $n \rightarrow \infty$ then $\langle f, \phi \rangle = \lim_{n \rightarrow \infty} \langle f, \phi_n \rangle$.

For any choice of a $L_{+, a, \lambda}$ is a subspace of E .

Moreover since D is dense in E and $D \subset L_{+, a, \lambda} \subset E$,

$L_{+, a, \lambda}$ is dense in E . Also we know the seminorms

for E are defined by $\gamma_{k, k}(\psi) = \sup_{t \in k} |D^k \psi(t)|$, $\psi \in E$.

where $k=0,1,2,3,\dots$

and k varies through the compact sets in \mathbb{R}' .

Let c be the infimum of $t^{k-\lambda} e^{at}$ on a given k ,

then c is a positive number.

Hence for any $\phi \in L_{+, a, \lambda}$,

$$\begin{aligned} \gamma_{+, a, k, \lambda}(\phi) &\geq \sup_{t \in k} |e^{at} t^{k-\lambda} D^k \phi(t)| \\ &\geq c \sup_{t \in k} |D^k \phi(t)| \geq c \gamma_{k, k}(\phi). \end{aligned}$$

Therefore topology of $L_{+, a, \lambda}$ is stronger than the topology

induced on $L_{+, a, \lambda}$ by E . We can immediately conclude that,

for every choice of w $L_{+, \lambda}(w)$ is a dense subspace of E ,

and that the convergence in $L_{+, \lambda}(w)$ implies the

convergence in E . Therefore E' is subspace of $L'_{+, a, \lambda}$

as well as $L'_{+, \lambda}(w)$.

Theorem :- For each $f \in L'_{+, a, \lambda}$ there exist a positive

constant c and a non negative integer r such that for all

$\phi \in L_{+, a, \lambda}$

$$|\langle f, \phi \rangle| \leq c \max_{0 < k < r} \gamma_{a, k, \lambda}(\phi) \dots \dots \dots (1)$$

Proof :- Let $\rho_{a, k, \lambda}(\phi) = \max_{0 < k < r} \gamma_{a, k, \lambda}$

Since $\gamma_{a, 0, \lambda}$ is norm $\rho_k(\phi)$ is also norm.

Assume that there are no values of c and r for which the

inequality (1) holds. This means that for each positive

integer $n \exists$ a $\phi_n \in L_{+, a, \lambda}$,

such that $|\langle f, \phi_n \rangle| > n \rho_{a, n, \lambda}(\phi_n) \dots \dots \dots (2)$

Since $\rho_{a, n, \lambda}(\phi)$ is norm $\rho_{a, n, \lambda}(\phi_n) > 0$

$$\text{Set } \theta_n = \frac{\phi_n}{n \rho_{a, n, \lambda}(\phi_n)} \in L_{+, a, \lambda}.$$

with k being an arbitrary but fixed non negative integer.

We have for $n > k$,

$$\rho_{a,k,\lambda}(\phi_n) \leq \rho_{a,n,\lambda}(\phi_n) = \frac{\rho_{a,n,\lambda}(\phi_n)}{n \rho_{a,n,\lambda}(\phi_n)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since topology generated by $\{\rho_{a,n,\lambda}\}$ and topology generated by $\{\gamma_{a,n,\lambda}\}$ are equal

it follows that $\{\theta_n\}$ converges to zero in $L_{+,a,\lambda}$.

Consequently $\langle f, \theta_n \rangle \rightarrow 0$, because f is continuous functional on $L'_{+,a,\lambda}$.

But from (2) we have $|\langle f, \theta_n \rangle| = \frac{|\langle f, \phi_n \rangle|}{n \rho_{a,n,\lambda}(\phi_n)} > 1$.

This is contradiction, which proves the theorem.

Theorem :- For every $\phi(t) \in D$, where D is a countable unionspace, $t^\lambda \phi(t) \in L_{+,a,\lambda}$ for all real λ ,

then $\phi(t) \rightarrow t^\lambda \phi(t)$ is a continuous linear mapping from D into $L_{+,a,\lambda}$.

Proof :- We consider

$$\begin{aligned} e^{at} t^{n-\lambda} D^n t^\lambda \phi(t) &= e^{at} t^{n-\lambda} \sum_{k=0}^n c_k D^{n-k} t^\lambda D^k \phi(t) \\ &\text{where } k=0,1,2,\dots \text{ \& } n=0,1,2,\dots \\ &= e^{at} t^{n-\lambda} \sum_{k=0}^n C_{n,k} t^{\lambda-n+k} D^k \phi(t) \\ &= \sum_{k=0}^n e^{at} C_{n,k} t^k D^k \phi(t). \end{aligned}$$

where $C_{n,k}$ are constants depending upon n & k .

Therefore $|e^{at} t^{n-\lambda} D^n t^\lambda \phi(t)| \leq M \sum_{k=0}^n |D^k \phi(t)| \dots\dots(1)$

where M is a constant.

But since $\phi(t) \in D$, so $\gamma_k(\phi) = \sup_{0 < t < \infty} |D^k \phi(t)|$ is bounded.

$$k=0,1,2,3,\dots$$

Thus from (1) we get,

$$\rho_n(t^\lambda \phi(t)) \leq M \sum_{k=0}^n \gamma_k(\phi) \dots\dots(2)$$

From (2) it follows that if $\phi(t) \in D$ then $t^\lambda \phi(t) \in L_{+,a,\lambda}$.

Obviously, $\phi(t) \rightarrow t^\lambda \phi(t)$ is a continuous & linear mapping of D into $L_{+,a,\lambda}$.

Hence $g(t) \rightarrow t^\lambda g(t)$ is a continuous linear mapping from $L'_{+,a,\lambda}$ into D' .

Thus we get $\langle t^\lambda g(t), \phi(t) \rangle = \langle g(t), t^\lambda \phi(t) \rangle \dots\dots\dots(3)$.

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