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Notes on Generalization of Vieta-Pell and Vieta-Pell Lucas polynomials

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Abstract

In this paper, we define a new generalization for Pell and Pell Lucas, and modified Pell sequences called generalized Vieta-Pell and Vieta-Pell -Lucas polynomial sequences. The Binet formulae, generating functions, sum formulas, differatation rules and some important properties for these sequences are given. And then we generate a matrix whose elements are of generalized Vieta-Pell terms. By using this matrix we derive some properties for generalized Vieta-Pell and generalized Vieta-Pell-Lucas polynomial sequences
Keywords: Vieta-Pell sequence, Vieta-Pell *Lucas* sequence, Binet formula, Generating functions.
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1. INTRODUCTION AND PRELIMINARIES

Special integer sequences especially Fibonacci sequence are encountered in different branches of science, art, nature, the structure of our body. Also, the Pell sequence is one of the most famous and curious numerical sequence in mathematics and have been widely studied from both algebraic and combinatorial prospectives. Generalizations of integer sequences were studied by researchers by different types, such as changing the initial conditions or adding new parameters to the recurrence relations, changing the recurrence relation with respect to parity of index n. For example Swammy in [4] defined the generalized Fibonacci and Lucas polynomials and their diagonal polynomials. Djordjevic considered the generating functions, explicit formulas for

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generalized Fibonacci and Lucas polynomials in [5]. A generalization of Fibonacci and Lucas numbers, the Fibonacci, Lucas polynomials are studied in [6] by Catalan and defined by the recurrence relations respectively $F_n(x) = xF_{n-1}(x)+F_{n-2}(x),\ F_0(x) = 0,\ F_1(x) = 1,\ l_n(x) = xl_{n-1}(x) + l_{n-2}(x),\ l_0(x) = 2,\ l_1(x) = x$. Also, A. Nalli, P. Haukkanen, in [7] gave some properties of generalized Fibonacci and Lucas polynomials. In [8,9], Uygun generalized Jacobsthal and Jacobsthal Lucas seqıences by various approaches. Our paper is about Pell and Pell Lucas sequences. So first of all we define the recurrence relations for Pell, Pell Lucas, medified Pell sequences as $P_n = 2P_{n-1} + P_{n-2},\ P_0 = 0,\ P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2},\ Q_0 = 2,\ Q_1 = 2$;and $q_n = 2q_{n-1} + q_{n-2},\ q_0 = 1,\ q_1 = 1$ for $n \geq 2$, respectively in [1,2]. A.F. Horadam and Bro. J.M. Mahon, defined Pell and Pell–Lucas polynomials in [10]. Halıcı gave ,some sums formulae for products of terms of Pell, Pell-Lucas and modified Pell sequences in[11]. Dasdemir, in [12] studied Pell, Pell-Lucas and modified Pell numbers by using marix. Halıcı and Akyüz, gave a different generalization for Pell sequences called bivariate Pell polynomials in [13]. In this study we are interested in vieta generalization of Pell, Pell-Lucas and modified Pell sequences. In literature, vieta Lucas polynomials were studied by Robbins in [14]. A.F.Horadam defined vieta polynomials in[15]. Vitula and Slota redefined vieta polynomials as modified Chebyshev polynomials in [16]. Taşcı and Yalçın gave Vieta-Pell and Vieta-Pell-Lucas polynomials in [17]. The authors studied generalization of vieta-Jacobsthal and Jacobsthal-Lucas polynomials in [18].

Pell and the Pell Lucas polynomial sequences sequences are defined recurrently by $P_n(x) = 2P_{n-1}(x) + P_{n-2}(x),\ P_0(x) = 0,\ P_1(x) = 1,\ n \geq 2$ and $Q_n(x) = 2Q_{n-1}(x) + Q_{n-2}(x),\ Q_0(x) = 2,\ Q_1(x) = 2,\ n \geq 2$ in [10].The object of this paper is to define a new generalization of Pell, Pell Lucas and modified Pell sequences by using polynomials as called generalized vieta Pell polynomial $P_{k,n}(x)$ and generalized vieta Pell Lucas polynomial $Q_{k,n}(x)$ and generalized modified vieta Pell polynomial $q_{k,n}(x)$. We record some basic properties, matrix form of $P_{k,n}(x)$, $Q_{k,n}(x)$, and $q_{k,n}(x)$.

2. THE GENERALIZED VIETA-PELL, PELL LUCAS, MODIFIED PELL POLYNOMIALS AND THEIR PROPERTIES

Definition 1. *Let $n \geq 2$ any integer. The generalized vieta-Pell polynomial $\{P_{k,n}(x)\}_{n\in\mathbb{N}}$ sequences are described by using the following recurrence relation*

$$P_{k,n}(x) = 2^k xP_{k,n-1}(x) - P_{k,n-2}(x) \tag{2.1}$$

with initial conditions are $P_{k,0}(x) = 0,\ P_{k,1}(x) = 1$,*and generalized vieta-Pell Lucas polynomial*$\{Q_{k,n}(x)\}_{n\in\mathbb{N}}$ *sequences are defined*

$$Q_{k,n}(x) = 2^k x Q_{k,n-1}(x) - Q_{k,n-2}(x) \tag{2.2}$$

with initial conditions are $Q_{k,0}(x) = 2,\ Q_{k,1}(x) = 2^k x$ *and generalized vieta modified-Pell polynomial* $\{q_{k,n}(x)\}_{n\in\mathbb{N}}$ *sequences are defined*

$$q_{k,n}(x) = 2^k x q_{k,n-1}(x) - q_{k,n-2}(x) \tag{2.3}$$

with initial conditions are $q_{k,0}(x) = 1,\ q_{k,1}(x) = 2^{k-1}x.$
Initially, the polynomials are defined for only positive terms but their existence for $n < 0$ *is readily obtained, yielding*

$$\begin{aligned} P_{k,-n}(x) &= -P_{k,n}(x), \\ Q_{k,-n}(x) &= Q_{k,n}(x), \\ q_{k,-n}(x) &= q_{k,n}(x). \end{aligned}$$

The first some terms of generalized vieta-Pell polynomials are $P_{k,1}(x) = 1,\ P_{k,2}(x) = 2^k x,\ P_{k,3}(x) = 2^{2k}x^2 - 1,\ P_{k,4}(x) = 2^{3k}x^3 - 2^{k+1}x.$
The first some terms of generalized vieta-Pell polynomial Lucas polynomials are $Q_{k,1}(x) = 2^k x,\ Q_{k,2}(x) = 2^{2k}x^2 - 2,\ Q_{k,3}(x) = 2^{3k}x^3 - 3.2^k x.$
The first some terms of generalized vieta modified-Pell polynomials are obtained by dividing into two the elements of generalized vieta Pell- Lucas polynomials.
The characteristic equation of recurrence relation for generalized vieta-Pell, Pell Lucas, and modified Pell polynomial sequences is

$$r^2 - 2^k x r + 1 = 0.$$

The roots of the characteristic equation are

$$\alpha(x) = \frac{2^k x + \sqrt{2^{2k}x^2 - 4}}{2}, \quad \beta(x) = \frac{2^k x - \sqrt{2^{2k}x^2 - 4}}{2} \tag{2.4}$$

with the following properties

$$\alpha(x) + \beta(x) = 2^k x, \quad \alpha(x) - \beta(x) = \sqrt{2^{2k}x^2 - 4}, \quad \alpha(x).\beta(x) = 1. \tag{2.5}$$

Lemma 2. *The Binet formulas for these sequences are*

$$P_{k,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \tag{2.6}$$

$$Q_{k,n}(x) = \alpha^n(x) + \beta^n(x), \tag{2.7}$$

$$q_{k,n}(x) = \frac{\alpha^n(x) + \beta^n(x)}{2}, \tag{2.9}$$

Proof. The proof is obtained easily by using the values of the first two terms of $P_{k,n}(x)$ and $Q_{k,n}(x)$, $q_{k,n}(x)$. □

(Important Relationships)

Important elementary relationships involving $P_{k,n}(x)$ and $Q_{k,n}(x)$ follow with the aid of (2.1)-(2.9)

$a)\ P_{k,n}(x)Q_{k,n}(x) = P_{k,2n}(x)$

$b)\ P_{k,n+1}(x) - P_{k,n-1}(x) = Q_{k,n}(x)$

$c)\ Q_{k,n+1}(x) - Q_{k,n-1}(x) = \left(\left(2^k x\right)^2 - 4\right) P_{k,n}(x)$

$d)\ Q_{k,n}(x)Q_{k,m}(x) = Q_{k,m+n}(x) + Q_{k,m-n}(x)$

$e)\ P_{k,n}(x)Q_{k,m}(x) = P_{k,n+m}(x) + P_{k,n-m}(x)$

$f)\ P_{k,n}(x)P_{k,m}(x) = \frac{1}{\left(\left(2^k x\right)^2 - 4\right)} \left(Q_{k,m+n}(x) - Q_{k,m-n}(x)\right)$

$g)\ Q_{k,n}^2(x) = \left(\left(2^k x\right)^2 - 4\right) P_{k,n}^2(x) + 4$

$h) Q_{k,n}^2(x) + \left(\left(2^k x\right)^2 - 4\right) P_{k,n}^2(x) = 2Q_{k,2n}(x)$

$i)\ Q_{k,n}^2(x) = Q_{k,2n}(x) + 2$

$j)\ P_{k,n+1}(x) - 2^{k-1}xP_{k,n}(x) = Q_{k,n}(x)/2$

$k)\ 2Q_{k,n+1}(x) - 2^k xQ_{k,n}(x) = \left(\left(2^k x\right)^2 - 4\right) P_{k,n}(x)$

$l)\ P_{k,n+1}^2(x) - P_{k,n}^2(x) = P_{k,2n+1}(x)$

$m)\ P_{k,n}^2(x) + P_{k,n+1}^2(x) = \frac{2^k xQ_{k,2n+1}(x) - 4}{\left(2^k x\right)^2 - 4}$

$n)\ Q_{k,n+1}^2(x) - Q_{k,n}^2(x) = \left(\left(2^k x\right)^2 - 4\right) P_{k,2n+1}$

$o)\ Q_{k,n+1}^2(x) + Q_{k,n}^2(x) = Q_{k,2n+2}(x) + Q_{k,2n}(x) + 4$

$p)\ Q_{k,n}(x)Q_{k,n+1}(x) + \left(\left(2^k x\right)^2 - 4\right) P_{k,n}(x)P_{k,n+1}(x) = 2Q_{k,2n+1}(x)$

$r)\ Q_{k,n}(x)Q_{k,n+1}(x) - \left(\left(2^k x\right)^2 - 4\right) P_{k,n}(x)P_{k,n+1}(x) = 2^{k+1}x$

$s)\ (Q_{k,n}(x))^{\gamma} = \sum_{i=0}^{\lfloor \frac{\gamma}{2} \rfloor} \binom{\gamma}{i} Q_{\gamma-2i}(x)$

$t)\ (P_{k,n}(x))^{\gamma} = \frac{\sum_{i=0}^{\lfloor \frac{\gamma}{2} \rfloor} \binom{\gamma}{t}(-1)^i\left[\xi(\gamma+1)Q_{k,\gamma-2i}(x)+\xi(\gamma)P_{\gamma-2i}(x)\right]}{\left(\left(2^k x\right)^2-4\right)^{\lfloor \frac{\gamma}{2} \rfloor}},$

where $\xi(\gamma) = \begin{cases} 1, & \gamma \text{ is odd} \\ 0, & \gamma \text{ is even} \end{cases}$

Theorem 3. ***(The generating function)*** *Let i, j any natural numbers and $|\alpha^i(x)t| < 1$ and $|\beta^i(x)t| < 1$. Then the generating functions of these sequences for different indices are obtained as*

$$\begin{aligned}
\sum_{n=0}^{\infty} P_{k,in+j}(x)t^n &= \frac{P_{k,j}(x) + P_{k,i-j}(x)t}{1 - Q_{k,i}(x)t + t^2}, \\
\sum_{n=0}^{\infty} Q_{k,in+j}(x)t^n &= \frac{Q_{k,j}(x) - Q_{k,i-j}(x)t}{1 - Q_{k,i}(x)t + t^2}, \\
\sum_{n=0}^{\infty} q_{k,in+j}(x)t^n &= \frac{q_{k,j}(x) + q_{k,i-j}(x)t}{(1 - Q_{k,i}(x)t + t^2)}.
\end{aligned}$$

Proof. By using Binet formula for generalized vieta-Pell polynomial *sequence, (2,6) we get*

$$\begin{aligned}
\sum_{n=0}^{\infty} P_{k,in+j}(x)t^n &= \sum_{n=0}^{\infty} \frac{\alpha^{in+j}(x) - \beta^{in+j}(x)}{\alpha(x) - \beta(x)} t^n \\
&= \frac{1}{\alpha(x) - \beta(x)} \left[\alpha^j \sum_{n=0}^{\infty} \left(\alpha^i t\right)^n - \beta^j \sum_{n=0}^{\infty} \left(\beta^i t\right)^n \right] \\
&= \frac{1}{\alpha(x) - \beta(x)} \left[\frac{\alpha^j}{1 - \alpha^i(x)t} - \frac{\beta^j}{1 - \beta^i(x)t} \right] \\
&= \frac{\left(\alpha^j(x) - \beta^j(x)\right) + \left(\alpha^{i-j}(x) - \beta^{i-j}(x)\right)t}{\left(\alpha(x) - \beta(x)\right)\left(1 - t\left(\alpha^i(x) + \beta^i(x)\right) + t^2\right)} \\
&= \frac{P_{k,j}(x) + P_{k,i-j}(x)t}{1 - Q_{k,i}(x)t + t^2}.
\end{aligned}$$

The other part of the proof is done by using the same method

Some examples *for different values of* i, j are given as

$$\begin{aligned}
\sum_{n=0}^{\infty} P_{k,n}(x)t^n &= \frac{t}{1-2^k xt+t^2},\\
\sum_{n=0}^{\infty} Q_{k,n}(x)t^n &= \frac{2-2^k xt}{1-2^k xt+t^2},\\
\sum_{n=0}^{\infty} q_{k,n}(x)t^n &= \frac{1-2^{k-1} xt}{1-2^k xt+t^2},\\
\sum_{n=0}^{\infty} P_{k,2n}(x)t^n &= \frac{2^k xt}{1-(2^{2k}x^2-2)t+t^2},\\
\sum_{n=0}^{\infty} Q_{k,2n}(x)t^n &= \frac{2-(2^{2k}x^2-2)t}{1-(2^{2k}x^2-2)t+t^2},\\
\sum_{n=0}^{\infty} q_{k,2n}(x)t^n &= \frac{1-(2^{2k-1}x^2-1)t}{1-(2^{2k}x^2-2)t+t^2},\\
\sum_{n=0}^{\infty} P_{k,2n+1}(x)t^n &= \frac{1-t}{1-(2^{2k}x^2-2)t+t^2},\\
\sum_{n=0}^{\infty} Q_{k,2n+1}(x)t^n &= \frac{Q_{k,1}(x)-Q_{k,1}(x)t}{1-(2^{2k}x^2-2)t+t^2},\\
\sum_{n=0}^{\infty} q_{k,2n+1}(x)t^n &= \frac{2^{k-1}x+2^{k-1}xt}{1-(2^{2k}x^2-2)t+t^2}.
\end{aligned}$$

□

Theorem 4. ***(Explicit closed form)*** *Let* $n \geq 1$

$$\begin{aligned}
P_{k,n}(x) &= 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} (2^k x)^{n-2i-1} \left(\left(2^{2k}x^2-4 \right) \right)^i\\
Q_{k,n}(x) &= 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (2^k x)^{n-2i} \left(2^{2k}x^2-4 \right)^i\\
q_{k,n}(x) &= 2^{-n}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (2^k x)^{n-2i} \left(2^{2k}x^2-4 \right)^i
\end{aligned}$$

Proof.

$$
\begin{aligned}
\alpha^n(x) - \beta^n(x) &= [(2^k x + \sqrt{2^{2k}x^2-4})^n - (2^k x - \sqrt{2^{2k}x^2-4})^n]/2^n \\
&= 2^{-n}\sum_{i=0}^{n} \binom{n}{i} (2^k x)^{n-i} \begin{bmatrix} \left(\sqrt{2^{2k}x^2-4}\right)^i \\ -\left(-\sqrt{2^{2k}x^2-4}\right)^i \end{bmatrix} \\
&= 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} (2^k x)^{n-2i-1} \left(\sqrt{2^{2k}x^2-4}\right)^{2i+1}
\end{aligned}
$$

$$
\begin{aligned}
P_{k,n}(x) &= \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \\
&= 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} (2^k x)^{n-2i-1} \left(2^{2k}x^2-4\right)^i
\end{aligned}
$$

$$
\begin{aligned}
\alpha^n(x) + \beta^n(x) &= [(2^k x + \sqrt{2^{2k}x^2-4})^n + (2^k x - \sqrt{2^{2k}x^2-4})^n]/2^n \\
&= 2^{-n}\sum_{i=0}^{n} \binom{n}{i} (2^k x)^{n-i} \begin{bmatrix} \left(\sqrt{2^{2k}x^2-4}\right)^i \\ +\left(-\sqrt{2^{2k}x^2-4}\right)^i \end{bmatrix} \\
&= 2^{-n+1}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} (2^k x)^{n-2i} \left(2^{2k}x^2-4\right)^i
\end{aligned}
$$

□

Theorem 5. *We can also find explicit closed form by using the generating function for $P_{k,n}(x)$, $Q_{k,n}(x)$ and $q_{k,n}(x)$. Let $n \geq 1$ any integer,*

$$P_{k,n}(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-j}{j} (-1)^j \left(2^k x\right)^{n-2j-1} \tag{2.10}$$

$$Q_{k,n}(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} (-1)^j \left(2^k x\right)^{n-2j} \tag{2.11}$$

$$q_{k,n}(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{2(n-j)} \binom{n-j}{j} (-1)^j \left(2^k x\right)^{n-2j} \tag{2.12}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} P_{k,n}(x) t^n &= \frac{t}{1-2^k x t + t^2} = t \sum_{n=0}^{\infty} \left(2^k x t - t^2\right)^n \\
&= t \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} \left(2^k x t\right)^{n-i} \left(-t^2\right)^i \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} \left(2^k x\right)^{n-i} (-1)^i t^{n+i+1} \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-i-1}{i} (-1)^i \left(2^k x\right)^{n-2i-1} t^n
\end{aligned}$$

From the equality of both sides, the desired result obtained. □

Theorem 6. ***(Sum Formulas)***

$$\begin{aligned}
\sum_{i=0}^{n-1} P_{k,i}(x) &= \frac{-P_{k,n}(x) + P_{k,n-1}(x) + 1}{2 - 2^k x}, \\
\sum_{i=0}^{n-1} Q_{k,i}(x) &= \frac{2 - Q_{k,1}(x) - Q_{k,n}(x) + Q_{k,n-1}(x)}{1 - 2^k x t + t^2}, \\
\sum_{i=0}^{n-1} q_{k,i}(x) &= \frac{1 - q_{k,1}(x) - q_{k,n}(x) + q_{k,n-1}(x)}{1 - 2^k x t + t^2},
\end{aligned}$$

Theorem 7. ***(Sum Formulas for Square of Terms)***

$$\begin{aligned}
\sum_{i=0}^{n-1} P_{k,i}^2 &= \frac{1}{\left(2^k x\right)^2 - 4} \left[\frac{-Q_{k,2n} + Q_{k,2n-2}}{2 - Q_{k,2}} - 2n + 1\right] \\
\sum_{i=0}^{n-1} Q_{k,i}^2 &= \left[\frac{-Q_{k,2n} + Q_{k,2n-2}}{2 - Q_{k,2}} + 2n + 1\right] \\
\sum_{i=0}^{n-1} q_{k,i}^2 &= \frac{1}{2} \left[\frac{-Q_{k,2n} + Q_{k,2n-2}}{2 - Q_{k,2}} + 2n + 1\right]
\end{aligned}$$

Proof.

$$\begin{aligned}
\sum_{i=0}^{n-1} P_{k,i}^2 &= \sum_{i=0}^{n-1} \left(\frac{\alpha^i - \beta^i}{\alpha - \beta}\right)^2 \\
&= \frac{1}{(\alpha-\beta)^2}\left[\sum_{i=0}^{n-1}\left(\alpha^{2i} + \beta^{2i} - 2\right)\right] \\
&= \frac{1}{(\alpha-\beta)^2}\left[\frac{\alpha^{2n}-1}{\alpha^2-1} + \frac{\beta^{2n}-1}{\beta^2-1} - 2n\right] \\
&= \frac{1}{(2^k x)^2 - 4}\left[\frac{-Q_{k,2n} + Q_{k,2n-2} + (2 - Q_{k,2})}{2 - Q_{k,2}} - 2n\right] \\
&= \frac{1}{(2^k x)^2 - 4}\left[\frac{-Q_{k,2n} + Q_{k,2n-2}}{2 - Q_{k,2}} - 2n + 1\right]
\end{aligned}$$

□

Corollary 8.

$$(\alpha-\beta)^2 \sum_{i=0}^{n-1} P_{k,i}^2 - \sum_{i=0}^{n-1} Q_{k,i}^2 = -4n$$

Theorem 9. *The derivates of* $P_{k,n}(x)$, $Q_{k,n}(x)$ *are obtained as in the following*

$$\frac{dP_{k,n}}{dx} = n2^{k-1}P_{k,n-1}(x) + \frac{2^{2k-3}x}{2^{2k-2}x^2 - 1}\left[nQ_{k,n-1} - 2P_{k,n}\right]$$

$$\frac{dQ_{k,n}(x)}{dx} = nP_{k,n}(x)$$

$$\frac{dQ_{k,n}(x)}{dx} = n2^{k-1}\left(Q_{k,n-1}(x) + 2^k x P_{k,n-1}(x)\right)$$

Proof. Let $b(x) = \sqrt{2^{2k-2}x^2 - 1}$ and $\frac{db(x)}{dx} = \frac{2^{2k-2}x}{\sqrt{2^{2k-2}x^2-1}} = \frac{2^{2k-2}x}{b(x)}$.

$$
\begin{aligned}
\frac{dP_{k,n}}{dx} &= \frac{1}{2}\frac{d\left(\frac{\left(2^{k-1}x+b(x)\right)^n+\left(2^{k-1}x-b(x)\right)^n}{b(x)}\right)}{d(x)} \\
&= \frac{1}{2b^2(x)}\left\{\left[n\left(2^{k-1}x+b(x)\right)^{n-1}\left(2^{k-1}+\frac{2^{2k-2}x}{b(x)}\right)\right.\right. \\
&\quad \left. -n\left(2^{k-1}x-b(x)\right)^{n-1}\left(2^{k-1}-\frac{2^{2k-2}x}{b(x)}\right)\right]b(x) \\
&\quad \left. -\left[\left(2^{k-1}x+b(x)\right)^n-\left(2^{k-1}x-b(x)\right)^n\right]\frac{2^{2k-2}x}{b(x)}\right]\Bigg\} \\
&= \frac{1}{2b^2(x)}\left\{nb^2(x)\,2^k\left[\frac{\left(2^{k-1}x+b(x)\right)^{n-1}-\left(2^{k-1}x-b(x)\right)^{n-1}}{2b(x)}\right]\right. \\
&\quad +n2^{2k-2}x\left[\left(2^{k-1}x+b(x)\right)^{n-1}+\left(2^{k-1}x-b(x)\right)^{n-1}\right] \\
&\quad \left. -2^{k-1}x\left[\frac{\left(2^{k-1}x+b(x)\right)^{n}-\left(2^{k-1}x-b(x)\right)^{n}}{2b(x)}\right]\right\} \\
&= \frac{1}{2b^2(x)}\left(nb^2(x)\,2^kP_{k,n-1}(x)+n2^{2k-2}xQ_{k,n-1}(x)-2^{k-1}xP_{k,n}(x)\right) \\
&= n2^{k-1}P_{k,n-1}(x)+\frac{2^{2k-3}x}{2^{2k-2}x^2-1}\left[nQ_{k,n-1}(x)-2^{1-k}P_{k,n}(x)\right]
\end{aligned}
$$

From (2.11), we get

$$
\begin{aligned}
\frac{dQ_{k,n}(x)}{dx} &= \left(\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n}{n-j}\binom{n-j}{j}(-1)^j\left(2^kx\right)^{n-2j}\right)' \\
&= \sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\frac{n(n-2j)}{n-j}\binom{n-j}{j}(-1)^j(2^kx)^{n-2j-1}2^k \\
&= \sum_{j=0}^{\lfloor \frac{n-1}{2}\rfloor}\frac{n(n-j-1)!}{(n-2j-1)!j!}(2^kx)^{n-2j-1}2^k \\
&= \sum_{j=0}^{\lfloor \frac{n-1}{2}\rfloor}\binom{n-1-j}{j}n(2^kx)^{n-1-2j}2^k
\end{aligned}
$$

For the last proof,

$$
\begin{aligned}
\frac{dQ_{k,n}(x)}{dx} &= \frac{d\left(\left(2^{k-1}x+b(x)\right)^{n}+\left(2^{k-1}x-b(x)\right)^{n}\right)}{dx} \\
&= n\left(2^{k-1}x+b(x)\right)^{n-1}\left[2^{k-1}+\frac{2^{2k-2}x}{b(x)}\right] \\
&\quad +n\left(2^{k-1}x-b(x)\right)^{n-1}\left[2^{k-1}-\frac{2^{2k-2}x}{b(x)}\right] \\
&= n2^{k-1}\left[\left(2^{k-1}x+b(x)\right)^{n-1}+\left(2^{k-1}x-b(x)\right)^{n-1}\right] \\
&\quad +\frac{n2^{2k-2}x}{b(x)}\left[\left(2^{k-1}x+b(x)\right)^{n-1}-\left(2^{k-1}x-b(x)\right)^{n-1}\right] \\
&= n2^{k-1}Q_{k,n-1}(x)+n2^{2k-1}x\left[\frac{\left(2^{k-1}x+b(x)\right)^{n-1}-\left(2^{k-1}x-b(x)\right)^{n-1}}{2b(x)}\right] \\
&= n2^{k-1}Q_{k,n-1}(x)+n2^{2k-1}xP_{k,n-1}(x) \\
&= n2^{k-1}\left[Q_{k,n-1}(x)+2^{k}xP_{k,n-1}(x)\right]
\end{aligned}
$$

□

Theorem 10. ***(D'ocagne's property)***

Let $n \geq m$ and $n, m \in \mathbb{Z}^{+}$. For generalized vieta-Pell and Pell *Lucas* polynomial *sequences*, we have

$$
\begin{aligned}
P_{k,m+1}(x)P_{k,n}(x)-P_{k,m}(x)P_{k,n+1}(x) &= P_{k,n-m}(x). \\
Q_{k,m+1}(x)Q_{k,n}(x)-Q_{k,m}(x)Q_{k,n+1}(x) &= -\sqrt{2^{2k}x^{2}-4}P_{k,n-m}(x). \\
q_{k,m+1}(x)q_{k,n}(x)-q_{k,m}(x)q_{k,n+1}(x) &= \tfrac{-1}{4}\sqrt{2^{2k}x^{2}-4}P_{k,n-m}(x).
\end{aligned}
$$

Proof. By using Binet formula, we have

$$
\begin{aligned}
& P_{k,m+1}(x)P_{k,n}(x)-P_{k,m}(x)P_{k,n+1}(x) \\
={}& \frac{\alpha^{m+1}(x)-\beta^{m+1}(x)}{\alpha(x)-\beta(x)}\frac{\alpha^{n}(x)-\beta^{n}(x)}{\alpha(x)-\beta(x)}-\frac{\alpha^{m}(x)-\beta^{m}(x)}{\alpha(x)-\beta(x)}\frac{\alpha^{n+1}(x)-\beta^{n+1}(x)}{\alpha(x)-\beta(x)} \\
={}& \frac{1}{(\alpha(x)-\beta(x))^{2}}\left[\begin{array}{c}\alpha^{n}(x)\beta^{m}(x)(\alpha(x)-\beta(x)) \\ -\alpha^{m}(x)\beta^{n}(x)(\alpha(x)-\beta(x))\end{array}\right] \\
={}& \frac{1}{\alpha(x)-\beta(x)}\left[(\alpha(x)\beta(x))^{m}(\alpha(x)^{n-m}-\beta(x)^{n-m})\right] \\
={}& \frac{1}{\alpha(x)-\beta(x)}\left[(\alpha(x)^{n-m}-\beta(x)^{n-m})\right].
\end{aligned}
$$

The D'ocagne's property for generalized vieta-Pell Lucas polynomial sequences can readily seen by using the same method. □

Theorem 11. ***(Catalan's property)***

Assume that $n, r \in \mathbb{Z}^{+}$. For generalized vieta Pell and Pell Lucas polynomial sequences, we have

$$\begin{aligned} P_{k,n+r}(x)P_{k,n-r}(x) - P_{k,n}^{2}(x) &= -P_{k,r}^{2}(x) \\ Q_{k,n+r}(x)Q_{k,n-r}(x) - Q_{k,n}^{2}(x) &= Q_{k,2r}(x) - 2. \\ q_{k,n+r}(x)q_{k,n-r}(x) - q_{k,n}^{2}(x) &= \tfrac{q_{k,2r}(x)-1}{2} \end{aligned}$$

Proof. The proof is readily obtained by Binet formula. □

Theorem 12. ***(Cassini's property or Simpson property)***

For $n \in \mathbb{Z}^{+}$, we have

$$\begin{aligned} P_{k,n+1}(x)P_{k,n-1}(x) - P_{k,n}^{2}(x) &= -1 \\ Q_{k,n+1}(x)Q_{k,n-1}(x) - Q_{k,n}^{2}(x) &= 2^{2k}x^{2} - 4 \\ q_{k,n+1}(x)q_{k,n-1}(x) - q_{k,n}^{2}(x) &= 2^{2k-2}x^{2} - 1. \end{aligned}$$

We get these properties by substituting 1 for r in Catalan property.

Theorem 13.

$$\begin{aligned} Q_{k,2n}(x) - 2 &= (2^{2k}x^{2} - 4)P_{k,n}^{2}(x) \\ -2Q_{k,2}(x) &= (2^{2k}x^{2} - 4)P_{k,n+1}(x)P_{k,n-1}(x) - Q_{k,n+1}(x)Q_{k,n-1}(x) \end{aligned}$$

Theorem 14. ***(Honsberg property)***

For $n \in \mathbb{Z}^{+}$, it's obtained that

$$P_{k,m+1}(x)P_{k,n+1}(x) + P_{k,m}(x)P_{k,n}(x) = \frac{Q_{k,m+n+2}(x) + Q_{k,m+n}(x) - 2Q_{k,m-n}(x)}{2^{2k}x^{2} - 4}$$

Theorem 15. *By this theorem we get new relations between the roots α, β and generalized vieta-Pell and Pell Lucas polynomial sequences.*

$$\begin{aligned} \alpha^{n}(x) &= \alpha(x)P_{k,n}(x) - P_{k,n-1}(x), \\ \beta^{n}(x) &= \beta(x)P_{k,n}(x) - P_{k,n-1}(x), \\ \sqrt{2^{2k}x^{2} - 4}\,\alpha^{n}(x) &= \alpha(x)Q_{k,n}(x) - Q_{k,n-1}(x), \\ -\sqrt{2^{2k}x^{2} - 4}\,\beta^{n}(x) &= \beta(x)Q_{k,n}(x) - Q_{k,n-1}(x). \end{aligned}$$

Proof. The proof is made by using Binet formula and the product of the roots,

$$\begin{aligned}\beta(x)P_{k,n}(x)-P_{k,n-1}(x) &= \beta(x)\frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}-\frac{\alpha^{n-1}(x)-\beta^{n-1}(x)}{\alpha(x)-\beta(x)}\\ &= \frac{1}{\alpha(x)-\beta(x)}\left[\beta(x)\left(\alpha^n(x)-\beta^n(x)\right)-\left(\alpha^{n-1}(x)-\beta^{n-1}(x)\right)\right]\\ &= \frac{1}{\alpha(x)-\beta(x)}\left(\alpha^{n-1}(x)-\beta^{n+1}(x)-\alpha^{n-1}(x)+\beta^{n-1}(x)\right)\\ &= \frac{1}{\alpha(x)-\beta(x)}\left[-\beta^{n-1}(x)\left(\beta^2(x)-1\right)\right]=\beta^n(x).\end{aligned}$$

Other proofs are done by using the same way. □

Theorem 16. *Assume that $A_0(x)=[0]$ and $A_n(x)$ is a nxn tridigional matrix defined as*

$$A_n(x)=\begin{bmatrix} 1 & -i & & & & \\ 0 & 2^kx & -i & . & & \\ & i & 2^kx & . & & \\ & & . & . & . & \\ & & & . & . & . \\ & & & & . & . & -i \\ & & & & & i & 2^kx \end{bmatrix}$$

where $i=\sqrt{-1}$ and $n\geq 0$. Then

$$\det A_n(x)=P_{k,n}(x).$$

Proof. The proof is made by mathematical induction method. For $n=0,1$, we have $\det A_0(x)=P_{k,0}(x)=0$ and $\det A_1(x)=P_{k,1}=1$. Assume that $\det A_{n-1}(x)=P_{k,n-1}(x)$, $\det A_n(x)=P_{k,n}(x)$ for $n>2$.

$$\det A_{n+1}(x)=2^kx\det A_n(x)+i^2\det A_{n-1}(x)=2^kxP_{k,n}(x)-P_{k,n-1}(x).$$

□

Theorem 17. *Assume that $B_n(x)$ is a nxn tridigional matrix defined as*

$$B_n(x)=\begin{bmatrix} 2 & -i & & & & \\ 0 & 2^kx/2 & -i & . & & \\ & i & 2^kx & . & & \\ & & . & . & . & \\ & & & . & . & . \\ & & & & . & . & -i \\ & & & & & i & 2^kx \end{bmatrix}$$

where $i=\sqrt{-1}$ *and* $n\geq 0$. *Then*

$$\det B_n(x)=Q_{k,n-1}(x).$$

Proof. The proof is obtained by using the proof of Theorem 16. □

3. MATRIX FORM OF GENERALIZED VIETA-PELL AND PELL LUCAS POLYNOMIAL SEQUENCES

We demonstrate that the matrix

$$P_{k,1}=\begin{bmatrix} 2^k x & 1 \\ -1 & 0 \end{bmatrix} \tag{3.1}$$

generates generalized vieta-Pell polynomials and generalized vieta-Pell Lucas polynomials. By using this matrix we can deduce some identities of these polynomials.

Theorem 18. *Let* n *is any positive integer. Then*

$$P_{k,1}^n=\begin{bmatrix} P_{k,n+1}(x) & P_{k,n}(x) \\ -P_{k,n}(x) & -P_{k,n-1}(x) \end{bmatrix} \tag{3.2}$$

Proof. For the proof, mathematical induction method is used. It is easily seen that the assertion is true for $n=1$. Assume that the statement is true for $m\leq n$. The result is also true for $n+1$. Because

$$\begin{aligned} P_{k,1}^{n+1}=P_{k,1}^n P_{k,1} &= \begin{bmatrix} P_{k,n+1}(x) & P_{k,n}(x) \\ -P_{k,n}(x) & -P_{k,n-1}(x) \end{bmatrix}\begin{bmatrix} 2^k x & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} P_{k,n+2}(x) & P_{k,n+1}(x) \\ -P_{k,n+1}(x) & -P_{k,n}(x) \end{bmatrix}. \end{aligned}$$

From this theorem we can write the following property for generalized vieta-Pell polynomials

$$\begin{bmatrix} P_{k,n+1}(x) \\ -P_{k,n}(x) \end{bmatrix}=P_{k,1}^n\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.3}$$

We can write a similar property for generalized vieta-Pell Lucas polynomials as the following:

$$\begin{bmatrix} Q_{k,n+1}(x) \\ -Q_{k,n}(x) \end{bmatrix}=P_{k,1}^n\begin{bmatrix} 2^k x \\ -2 \end{bmatrix} \tag{3.4}$$

□

Corollary 19. *Let m, n are any positive integers, then*

$$P_{k,m+n+1}(x) = P_{k,n+1}(x)P_{k,m+1}(x) - P_{k,n}(x)P_{k,m}(x)$$

Proof. The proof is made by using the property of $P_{k,1}^{m+n} = P_{k,1}^{m}.P_{k,1}^{n}$ and equality of matrix. □

Corollary 20.

$$\begin{aligned} P_{k,m+n}(x) &= P_{k,n+1}(x)P_{k,m}(x) - P_{k,n}(x)P_{k,m-1}(x) \\ Q_{k,m+n}(x) &= Q_{k,n+1}(x)P_{k,m}(x) - Q_{k,n}(x)P_{k,m-1}(x) \end{aligned}$$

Proof. We also can see the truth of the relation by using the Corollory 19. We want to use another method for the proof by using (3.3), (3.4).

$$\begin{aligned} P_{k,n+1}(x)P_{k,m}(x) - P_{k,n}(x)P_{k,m-1}(x) &= \begin{bmatrix} P_{k,m}(x) & P_{k,m-1}(x) \end{bmatrix} \begin{bmatrix} P_{k,n+1}(x) \\ -P_{k,n}(x) \end{bmatrix} \\ &= \begin{bmatrix} P_{k,m}(x) & P_{k,m-1}(x) \end{bmatrix} P_{k,1}^{n} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} P_{k,1}^{m+n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= P_{k,m+n}(x) \end{aligned}$$

and

$$\begin{aligned} & P_{k,m}(x)Q_{k,n+1}(x) + Q_{k,n}(x)P_{k,m-1}(x) \\ = & \begin{bmatrix} P_{k,m}(x) & P_{k,m-1}(x \end{bmatrix} \begin{bmatrix} Q_{k,n+1}(x) \\ -Q_{k,n}(x) \end{bmatrix} \\ = & \begin{bmatrix} P_{k,m}(x) & P_{k,m-1}(x \end{bmatrix} P_{k,1}^{n} \begin{bmatrix} 2^k x \\ -2 \end{bmatrix} \\ = & \begin{bmatrix} 1 & 0 \end{bmatrix} P_{k,1}^{m+n-1} \begin{bmatrix} 2^k x \\ -2 \end{bmatrix} \\ = & Q_{k,m+n}(x) \end{aligned}$$

□

Corollary 21. *Let $n \geq 0$, then*

$$P_{k,n+1}(x)P_{k,n-1}(x) - P_{k,n}^{2}(x) = -1.$$

Proof. $\det(P_{k,1}) = 1$. Then $\det(P_{k,1}^n) = 1 = -P_{k,n+1}(x)P_{k,n-1}(x) + P_{k,n}^2(x)$. □

Theorem 22. *Assume that A is a square matrix such that $A^2 = 2^k xA - I$. Then for all positive integers*

$$A^n = P_{k,n}(x)A - P_{k,n-1}(x)I.$$

Proof. The proof is obvious from induction. A can be chosen $P_{k,1}$. I is unit matrix. □

Theorem 23. *The inverse of $P_{k,n}$*

$$P_k^{-n} = \begin{bmatrix} -P_{k,n-1}(x) & -P_{k,n}(x) \\ P_{k,n}(x) & P_{k,n+1}(x) \end{bmatrix}$$

Theorem 24. *The eigenvalues of $P_{k,n}$ are $\alpha^n(x)$ and $B^n(x)$.*

Proof.

$$\begin{aligned} \det(P_k^n - \lambda I) &= \det\left(\begin{bmatrix} P_{k,n+1}(x) - \lambda & P_{k,n}(x) \\ -P_{k,n}(x) & -P_{k,n-1}(x) - \lambda \end{bmatrix}\right) = 0 \\ &= \lambda^2 - \lambda(P_{k,n+1}(x) - P_{k,n-1}(x)) - P_{k,n+1}(x)P_{k,n-1}(x) + P_{k,n}^2(x) \\ &= \lambda^2 - \lambda Q_{k,n}(x) + 1 \end{aligned}$$

The roots are found from Theorem 15.

$$\begin{aligned} \alpha^n(x) &= \frac{Q_{k,n}(x) + \sqrt{2^{2k}x^2 - 4}P_{k,n}(x)}{2} \\ \beta^n(x) &= \frac{Q_{k,n}(x) - \sqrt{2^{2k}x^2 - 4}P_{k,n}(x)}{2} \end{aligned}$$

□

Corollary 25. *We offer two relationships that can be described as being of the* Moivre type

$$\begin{aligned} \left[\sqrt{2^{2k}x^2 - 4}P_{k,n}(x) + Q_{k,n}(x)\right]^r &= 2^r \alpha^{nr} \\ &= 2^{r-1}\left[\sqrt{2^{2k}x^2 - 4}P_{k,nr}(x) + Q_{k,nr}(x)\right] \\ \left[\sqrt{2^{2k}x^2 - 4}P_{k,n}(x) - Q_{k,n}(x)\right]^r &= (-2)^r \beta^{nr} \\ &= (-2)^{r-1}\left[\sqrt{2^{2k}x^2 - 4}P_{k,nr}(x) - Q_{k,nr}(x)\right]. \end{aligned}$$

When $k = x = 1$ this property reduces to the following form

$$\begin{aligned} \left[\frac{Q_n}{2}\right]^r &= \frac{Q_{nr}}{2}, \\ \left[\frac{-Q_n}{2}\right]^r &= \frac{(-1)^r Q_{nr}}{2}. \end{aligned}$$

Conclusion 26. *In this study we obtained some basic and interesting properties of a new generalization of Pell and Pell Lucas and modified Pell sequences called generalized vieta Pell and Pell Lucas and modified Pell sequences.*

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