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Haar and Shannon Wavelets on Certain Locally Compact Abelian Groups

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Abstract

Haar and Shannon wavelets are the simple and fundamental wavelets on $L^2(\mathbb{R})$. On a Euclidean domain, both the wavelets are at poles apart in the way because, Haar wavelet basis is well localized in space but not in frequency in contrast to Shannon wavelet basis which is well localized in frequency domain but not in time domain. In this paper, Haar and Shannon wavelets are constructed on a locally compact Abelian group G having open compact subgroup, using multiresolution analysis and established that both are same on G. Also, the proposed construction is illustrated using examples of locally compact Abelian groups.
Keywords: Wavelet; expansive automorphism; multiresolution analysis; Cantor Dyadic group; Vilenkin's group; p-adic field.

1. INTRODUCTION

Wavelet analysis is a recent development in the area of pure and applied Mathematics. It is an exciting new problem-solving tool with a wide range of applications. The main goal of wavelet theory is to create a collection of orthonormal basis functions which are used to describe a signal. Alfréd Haar was the first scientist to construct a wavelet.[2] Then a pool of mathematicians including J. Morlet, A. Grossmann, Y. Meyer, S. Mallat, I. Daubechies and many others have developed different wavelets.[7,11,17,18] The concept of wavelets is then extended to $\mathbb{R}^n$ as well as many other topological groups such as locally compact Abelian group.[1,3,4,6,9,14,15]

There are two methods to generate a wavelet system, one is using multiresolution analysis and other is using wavelet sets, in which multiresolution analysis is a key ingredient in this field. The multiresolution decomposition seperates components of a signal in a way, that is superior to all other methods. Wavelets are better signal representations because of this powerful and flexible decomposition. This is the motivation behind the application of wavelet transforms in a vast number of areas including molecular dynamics, astrophysics, quantum mechanics, signal processing, DNA analysis, climatology, image processing, speech recognition, computer graphics, etc. Michael W Frazier[10] explained the method for construction of multiresolution analysis and MRA wavelets.

Haar and Shannon wavelet are the basic wavelets on $L^2(\mathbb{R})$. Haar wavelet is a compactly supported step function in time. It is well localized in space but not frequency localized. On the other hand, Shannon wavelet is a compactly supported step function in frequency but not localized in time. That is, Haar wavelet is entirely different from Shannon wavelet on $L^2(\mathbb{R})$. But on a locally compact Abelian group both are same and this surprising fact is proved by R. L. Benedetto using wavelet sets.[5]

M. Skopina and E. Lebedeva[16] discussed about the Haar basis and it's multiresolution analysis on Cantor Dyadic group. In,[19] the authors presented the Shannon type orthonormal multiwavelet on a local field of positive characteristic. Yu. A. Farkov[8] analysed the basic constructions of MRA-based orthogonal wavelets and tight frames in Walsh analysis. Also, he mentioned the similarity of Haar and Shannon wavelets.

Since both Haar and Shannon wavelets are MRA wavelets, one can construct multiresolution analysis corresponding to them. It is much easier to deal with wavelets in time domain than frequency domain. Since mutiresolution decomposition is more flexible, it is important to construct multiresolution analysis of these wavelets on a locally compact Abelian group. This paper is devoted to the construction of Haar and Shannon wavelets on locally compact Abelian groups having open compact subgroups using multiresolution analysis. Section 1 contains the basic properties and important theorems on a locally compact Abelian group G. Section 2 deals with the construction of Haar wavelet and the multiresolution analysis corresponding to it. Also, it is proved that Haar wavelet is same as Shannon wavelet on a locally compact Abelian group having compact open subgroup. Illustration of the developed theory is presented in section 3 through important examples like Cantor Dyadic group, Vilenkin's group and p-adic field.

2. PRELIMINARIES

Let G be a locally compact Abelian group and $\widehat{G}$ be the dual group of G. Let μ and ν are the Haar measures defined on G and $\widehat{G}$, repectively. Denote the action of $\xi \in \widehat{G}$ on $x \in G$ by $\chi(x, \xi)$.

The annihilator[12] of a nonempty subgroup K of G in $\widehat{G}$ is given by

$$K^{\perp} = \{\gamma \in \widehat{G} : \chi(x, \gamma) = 1, \ \forall x \in K\}.$$

Let A be an automorphism of G. The adjoint of A of G is an automorphism of $\widehat{G}$ satisfying

$$\chi(Ax, \xi) = \chi(x, A^*\xi).$$

The modulus of A is given by

$$|A| = \frac{\mu(AU)}{\mu(U)},$$

for any compact subset U of G with positive measure. Then $|A^*| = |A|$.

Let K be an open compact subgroup of G. Normalize the Haar measure on G and $\widehat{G}$ so that $\mu(K) = 1$ and $\nu(K^\perp) = 1$. Label the open compact subgroup K as the fundamental domain in G.

Definition 2.1. [5] Let G be a locally compact abelian group and K be the fundamental domain in G. An automorphism A of G is said to be expansive with respect to K if,

1. $AK \supsetneq K$
2. $\cap_{n\leq 0} A^n K = \{0\}$.

Since $\mu(K) = 1$, $|A| = \mu(AK) > 0$ and hence $d\mu(Ax) = |A| d\mu(x)$. Similarly, since $\nu(K^\perp) = 1$, $|A| = |A^*| = \nu(A^*K^\perp)$, so that $d\nu(A^*\xi) = |A| d\nu(\xi)$.

Definition 2.2. [5] Let G be a locally compact abelian group. The operator $D : L^2(G) \to L^2(G)$ defined by,

$$D(f)(x) = |A|^{1/2} f(Ax)$$

is called the **dilation operator** on $L^2(G)$.

Definition 2.3. [5] Let G be a locally compact abelian group and C be the collection of coset representatives for the quotient group $\widehat{G}/K^\perp$. Define functions $\alpha : \widehat{G} \to C$ and $\beta : \widehat{G} \to K^\perp$ by

$$\alpha(\xi) = \text{ the unique } \sigma \in C \text{ such that } \xi - \sigma \in K^\perp, \text{ and}$$

$$\beta(\xi) = \xi - \alpha(\xi).$$

For any $[s] \in G/K$, let $\omega_{[s]}(\xi) = \overline{\chi(s, \beta(\xi))}$. Define $\tilde{\tau}_{[s]} : L^2(\widehat{G}) \to L^2(\widehat{G})$ by,

$$\tilde{\tau}_{[s]}(\widehat{f})(\xi) = \omega_{[s]}(\xi)\widehat{f}(\xi),$$

where $\widehat{f}$ is the Fourier transform of f. Then the translation operator $\tau_{[s]}$ on $L^2(G)$ can be defined as:

$$\tau_{[s]}(f)(x) = (\omega_{[s]} * \widehat{f})(x), \ x \in G.$$

Remark 2.1. 1. $|\omega_{[s]}(\xi)| = 1, \ \forall \xi \in \widehat{G}.$

2. For $t \in K, \quad \omega_{[t]}(\xi) = 1, \ \forall \xi \in \widehat{G}.$

3. $\omega_{[s+t]}(\xi) = \overline{\chi(s+t, \eta(\xi))} = \overline{\chi(s, \eta(\xi))\chi(t, \eta(\xi))} = \omega_{[s]}(\xi)\omega_{[t]}(\xi).$

Definition 2.4. Let G be a locally compact Abelian group and K be the fundamental domain in G. Let A be an expansive automorphism of G. Then a multiresolution analysis of $L^2(G)$ is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(G)$ such that,

1. $\cdots V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots$
2. $\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(G)$ and $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$.
3. $f(x) \in V_j \Leftrightarrow f(A(x)) \in V_{j+1}$.
4. $f(x) \in V_0 \Leftrightarrow \tau_{[s]} f(x) \in V_0, \ \forall\, [s] \in G/K.$
5. There exists $\phi \in L^2(G)$ so that $\{\tau_{[s]}\phi\}_{[s]\in G/K}$ is an orthonormal basis for V_0.

Since contractive automorphisms are special cases of expansive automorphisms, the theorems in[13] can be restated as follows :

Theorem 2.1. *Let G be a locally compact Abelian group with fundamental domain K and A be an expansive automorphism of G. Let $\phi \in L^2(G)$ and $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed subspaces of $L^2(G)$, where $V_0 = \overline{span}\{\tau_{[s]}\phi : [s] \in G/K\}$, $V_j = D^j V_0, \ j \in \mathbb{Z}$. Then*

$$\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(G) \Leftrightarrow \cup_{j\in\mathbb{Z}} \,(supp\, \widehat{\phi}_j) = \widehat{G}, \ a.e.,$$

where $\phi_j = D^j\phi, \ j \in \mathbb{Z}$.

Theorem 2.2. *Let G be a nondiscrete locally compact Abelian group with fundamental domain K. Let $\phi \in L^2(G)$ and $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed subspaces of $L^2(G)$, where*

$$V_0 = \overline{span}\,\{\tau_{[s]}\phi : [s] \in G/K\}, \ V_j = D^j V_0, \ j \in \mathbb{Z}.$$

Suppose that $\{\tau_{[s]}\phi : [s] \in G/K\}$ is an orthonormal basis for V_0. Then

$$\cap_{j\in\mathbb{Z}} V_j = \{0\}.$$

Theorem 2.3. *Let G be a locally compact Abelian group with fundamental domain K and A be an expansive automorphism of G. Let $\phi \in L^2(G)$ and $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed subspaces of $L^2(G)$, where $V_0 = \overline{span}\{\tau_{[s]}\phi : [s] \in G/K\}$, $V_j = D^j V_0, \ j \in \mathbb{Z}$. Suppose that $\{\tau_{[s]}\phi : [s] \in G/K\}$ is an orthonormal basis for V_0. Then $V_j \subseteq V_{j+1}$ if and only if ϕ is a scaling function.*

3. HAAR AND SHANNON WAVELETS

In Euclidean space Haar and Shannon wavelets are entirely different. In this work, the authors proved that both the wavelets are same on a locally compact Abelian group with a fundamental domain by constructing the corresponding multiresolution analysis.

Theorem 3.1. *Let G be a locally compact Abelian group and K be the fundamental domain in G. Let A be an expansive automorphism of G and N be the modulus of A. Define ϕ to be the indicator function of K. That is,*

$$\phi = 1_K.$$

Then ϕ is a scaling function with mask

$$m_0(\xi) = \frac{1}{N}\left(1 + \sum_{j=1}^{N-1} \omega_{[k_j]}(\xi)\right).$$

Proof. We have

$$\begin{aligned}\widehat{\phi}(\gamma) &= \int_G \phi(x)\overline{\chi(x,\gamma)}d\mu(x) \quad (3.1)\\ &= \int_K \overline{\chi(x,\gamma)}d\mu(x) \quad (3.2)\\ &= 1_{K^\perp}. \quad (3.3)\end{aligned}$$

Since A is expansive, $AK \supset K$.

Let AK/K has N distinct coset representatives $\{k_0, k_1, \ldots k_{N-1}\}$ with $k_0 = 0 \in K$. That is, $(k_i + K) \cap (k_j + K) = \emptyset$ and $AK = \cup_{j=0}^{N-1}(k_j + K)$.

For $x \in K$, $Ax \in k_j + K$ for some $j = 0, 1, \ldots N-1$. Then $Ax - k_j \in K$ and $Ax - k_l \notin K$ for $l \neq j$.

That is, $\phi(Ax - k_j) = 1$ and $\phi(Ax - k_l) = 0, \ \forall l \neq j$. Thus we have,

$$\phi(x) = \sum_{j=0}^{N-1} \phi(Ax - k_j).$$

Then

$$\begin{aligned}
\widehat{\phi}(\xi) &= \int_G \sum_{j=0}^{N-1} \phi(Ax - k_j)\overline{\chi(x,\xi)}d\mu(x) \\
&= \sum_{j=0}^{N-1} \int_G \phi(Ax - k_j)\overline{\chi(x,\xi)}d\mu(x) \\
&= \sum_{j=0}^{N-1} \int_G \phi(x - k_j)\overline{\chi(A^{-1}x,\xi)}d\mu(A^{-1}x) \\
&= \sum_{j=0}^{N-1} \int_G \frac{1}{N}\phi(x - k_j)\overline{\chi(x,(A^*)^{-1}\xi)}d\mu(x) \qquad (3.4) \\
&= \frac{1}{N}\sum_{j=0}^{N-1} \widehat{\tau_{[k_j]}\phi}((A^*)^{-1}\xi) \\
&= \frac{1}{N}\sum_{j=0}^{N-1} \omega_{[k_j]}(\xi)\widehat{\phi}((A^*)^{-1}\xi) \\
&= \frac{1}{N}\left(1 + \sum_{j=1}^{N-1} \omega_{[k_j]}(\xi)\right)\widehat{\phi}((A^*)^{-1}\xi),
\end{aligned}$$

since $k_0 = 0, \omega_{[k_0]} = 1$. □

Definition 3.1. Let

$$V_0 = \overline{span}\{\tau_{[s]}\phi : [s] \in G/K\},$$

and define the sequence $\{V_j\}_{j\in\mathbb{Z}}$ by, $f(\cdot) \in V_j \Leftrightarrow f(A^{-j}\cdot) \in V_0$.

Then $\{V_j\}_{j\in\mathbb{Z}}$ satisfies the conditions,

$$f(g) \in V_j \Leftrightarrow f(A(g)) \in V_{j+1},$$

and

$$f(g) \in V_0 \Leftrightarrow \tau_{[s]}f(g) \in V_0, \;\; \forall\, [s] \in G/K.$$

The following two theorems proves that $\{V_j\}_{j\in\mathbb{Z}}$ is a multiresolution analysis for $L^2(G)$.

Theorem 3.2. *Let G be a locally compact Abelian group and K be the fundamental domain in G. Let $\phi = 1_K$ and $V_0 = \overline{span}\{\tau_{[s]}\phi : [s] \in G/K\}$. Then for any $[s] \in G/K$, the translation of ϕ by $[s]$ is,*

$$\tau_{[s]}\phi(x) = 1_{s+K}(x).$$

Moreover, $\{\tau_{[s]}\phi : [s] \in G/K\}$ is an orthonormal basis for V_0.

Proof. Here $\widehat{\phi} = 1_{K^\perp}$, so that,

$$\tilde{\tau}_{[s]}\widehat{\phi}(\xi) = \overline{\chi(s, \beta(\xi))}\widehat{\phi}(\xi).$$

Also, for $\xi \in K^\perp$, $\beta(\xi) = \xi$.

$$\begin{aligned} \tau_{[s]}\phi(x) &= \int_{\widehat{G}} \tilde{\tau}_{[s]}\widehat{\phi}(\xi)\chi(x, \xi)d\nu(\xi) \\ &= \int_{K^\perp} \overline{\chi(s, \xi)}\chi(x, \xi)d\nu(\xi) \\ &= \int_{K^\perp} \overline{\chi(x - s, \xi)}d\nu(\xi) \\ &= 1_{s+K}(x). \end{aligned} \tag{3.5}$$

Since $s + K$ are disjoint, $\{\tau_{[s]}\phi\}$ is an orthogonal set in $L^2(G)$.
Now $\|\tau_{[s]}\| = 1$. Hence $\{\tau_{[s]}\phi : [s] \in G/K\}$ is an orthonormal basis for V_0. □

Theorem 3.3. *Let G be a locally compact Abelian group and K be the fundamental domain in G. Let A be an expansive automorphism of G. Define $\phi = 1_K$ and $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed subspaces of $L^2(G)$, where $V_0 = \overline{span}\{\tau_{[s]}\phi : [s] \in G/K\}$, $V_j = D^jV_0$, $j \in \mathbb{Z}$. Then the sequence $\{V_j\}_{j\in\mathbb{Z}}$ form a multiresolution analysis of $L^2(G)$.*

Proof. From the definition of the V_j and using Theorem 3.2, $\{V_j\}_{j\in\mathbb{Z}}$ satisfies the conditions 3,4, and 5 of multiresolution analysis. From Theorem 2.3 and 3.1, we have

$$\cdots V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots$$

Using Theorem 2.2 and 3.2,

$$\cap_{j\in\mathbb{Z}} V_j = \{0\}.$$

Now it remains to prove that

$$\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(G).$$

By Theorem 2.1, it is enough to show that

$$\cup_{j\in\mathbb{Z}}\,(supp\,\widehat{\phi}_j) = \widehat{G},\ a.e.,$$

where $\phi_j = D^j\phi,\ j \in \mathbb{Z}$.
Here $\phi = 1_K$ and $D(f) = |A|^{\frac{1}{2}}f(Ax)$. So,

$$\phi_j(x) = D^j\phi(x) = |A|^{\frac{j}{2}}\phi(A^jx) = |A|^{\frac{j}{2}}1_K(A^jx).$$

Then

$$\begin{aligned}\widehat{\phi}_j(\xi) &= \int_G |A|^{\frac{j}{2}} 1_K(A^j x)\overline{\chi(x,\xi)}d\mu(x)\\ &= |A|^{\frac{j}{2}} \int_G 1_K(x)\overline{\chi(A^{-j}x,\xi)}d\mu(A^{-j}x)\\ &= |A|^{\frac{-j}{2}} \int_K \overline{\chi(x,(A^*)^{-j}\xi)}d\mu(x)\\ &= |A|^{\frac{-j}{2}} 1_{K^\perp}((A^*)^{-j}\xi)\\ &= |A|^{\frac{-j}{2}} 1_{(A^*)^j K^\perp}(\xi).\end{aligned} \tag{3.6}$$

Hence $supp\,\widehat{\phi}_j = (A^*)^j K^\perp$. Therefore, $\cup_{j\in\mathbb{Z}}\,(supp\,\widehat{\phi}_j) = \cup_{j\in\mathbb{Z}}(A^*)^j K^\perp = \widehat{G}$. □

Remark 3.1. Since the scaling function defined in Theorem 3.1 is the characteristic function of fundamental domain, it is the scaling function associated with Haar wavelet. Thus Theorem 3.3 yields the multiresolution analysis corresponding to Haar scaling function. The following theorem explains a novel method for construction of wavelet corresponding to the multiresolution analysis.

Theorem 3.4. *Let G be a locally compact Abelian group and K be the fundamental domain in G. Let A be an expansive automorphism of G and N be the modulus of A. Let $\phi = 1_K$ so that $\widehat{\phi}(\xi) = m_0((A^*)^{-1}\xi)\widehat{\phi}((A^*)^{-1}\xi)$. Let $\{\sigma_0,\, \sigma_1,\, \ldots \sigma_{N-1}\}$ be the distinct coset representatives of $A^*K^\perp/K^\perp$ with $\sigma_0 \in K^\perp$. Define $\psi_j,\ j = 1, 2, \ldots N-1$ as,*

$$\widehat{\psi_j} = 1_{\sigma_j + K^\perp}.$$

Then

$$\widehat{\psi_j}(\xi) = m_j((A^*)^{-1}\xi)\widehat{\phi}((A^*)^{-1}\xi),$$

where $\sum_{j=0}^{N-1} m_j(\xi) = 1$.

Moreover, $\{\psi_1, \psi_2, \ldots \psi_{N-1}\}$ is the wavelet system for $L^2(G)$ corresponding to ϕ.

Proof. Here,

$$\begin{aligned}\psi_j(x) &= \int_{\widehat{G}} \widehat{\psi_j}(\xi)\chi(x,\xi)d\nu(\xi)\\ &= \int_{\sigma_j+K^\perp} \chi(x,\xi)d\nu(\xi)\\ &= \int_{K^\perp} \chi(x,\sigma_j+\xi)d\nu(\xi)\\ &= \int_{K^\perp} \chi(x,\sigma_j)\chi(x,\xi)d\nu(\xi)\\ &= \chi(x,\sigma_j)\int_{K^\perp} \chi(x,\xi)d\nu(\xi)\\ &= \chi(x,\sigma_j)1_K.\end{aligned} \tag{3.7}$$

Let AK/K has N distinct coset representatives $\{k_0, k_1, \ldots k_{N-1}\}$ with $k_0 = 0 \in K$. Define $B_l = A^{-1}(k_l + K)$. Then

$$K = \cup_{l=0}^{N-1} B_l \text{ and } B_l \cap B_j = \emptyset, \text{ for } l \neq j.$$

Also, for each $x \in B_l$, $\chi(x, \sigma_j) = c_{lj}$, a constant does not depand on x, such that $c_{00} = 1$. For fixed l, c_{lj} are N^{th} roots of unity.
Then we have,

$$\begin{aligned} \psi_j(x) &= c_{0j}\phi(Ax) + c_{1j}\phi(Ax - k_1) + \cdots + c_{(N-1)j}\phi(Ax - k_{N-1}) \\ &= \sum_{l=0}^{N-1} c_{lj}\phi(Ax - k_l). \end{aligned} \tag{3.8}$$

$$\begin{aligned} \widehat{\psi_j}(\xi) &= \int_G \sum_{l=0}^{N-1} c_{lj}\phi(Ax - k_l)\overline{\chi(x,\xi)}d\mu(x) \\ &= \sum_{l=0}^{N-1} c_{lj} \int_G \phi(Ax - k_l)\overline{\chi(x,\xi)}d\mu(x) \\ &= \sum_{l=0}^{N-1} c_{lj} \int_G \phi(x - k_l)\overline{\chi(A^{-1}x,\xi)}d\mu(A^{-1}x) \\ &= \sum_{l=0}^{N-1} c_{lj} \int_G \frac{1}{N}\phi(x - k_l)\overline{\chi(x,(A^*)^{-1}\xi)}d\mu(x) \\ &= \frac{1}{N}\sum_{l=0}^{N-1} c_{lj}\widehat{\tau_{[k_l]}\phi}((A^*)^{-1}\xi) \\ &= \frac{1}{N}\sum_{l=0}^{N-1} c_{lj}\omega_{[k_l]}(\xi)\widehat{\phi}((A^*)^{-1}\xi) \\ &= \frac{1}{N}\left(1 + \sum_{l=1}^{N-1} c_{lj}\omega_{[k_l]}(\xi)\right)\widehat{\phi}((A^*)^{-1}\xi). \end{aligned} \tag{3.9}$$

Thus $m_j(\xi) = \frac{1}{N}\left(1+\sum_{l=1}^{N-1} c_{lj}\omega_{[k_l]}(\xi)\right)$. Now

$$\begin{aligned}\sum_{j=0}^{N-1} m_j(\xi) &= \frac{1}{N}\left(1+\sum_{l=1}^{N-1}\omega_{[k_l]}(\xi)\right)+\frac{1}{N}\left(1+\sum_{l=1}^{N-1}c_{l1}\omega_{[k_l]}(\xi)\right)+\cdots\\ &\quad+\frac{1}{N}\left(1+\sum_{l=1}^{N-1}c_{l(N-1)}\omega_{[k_l]}(\xi)\right)\\ &= \frac{1}{N}\left[N+\sum_{l=1}^{N-1}\omega_{[k_l]}(\xi)\left(\sum_{j=0}^{N-1}c_{lj}\right)\right]\\ &= 1,\end{aligned} \tag{3.10}$$

since $\sum_{j=0}^{N-1} c_{lj} = 0$. □

Proposition 3.1. *Let G be a locally compact Abelian group having fundamental domain. The Haar and Shannon wavelets can be constructed on G. Moreover, both the wavelets are same on G.*

Proof. By Theorem 3.1, the scaling function is the indicator function of compact subset K of G. Also, by Theorem 3.4, each ψ_j is constant on each B_l, $l = 0, 1, \ldots N-1$, disjoint subsets of K. Therefore, the functions ψ_j constitute the Haar wavelets on G.

On the other hand, by Theorem 3.4, the Fourier transform of ψ_j is the indicator function of $\sigma_j + K^{\perp}$. Thus, the functions ψ_j also forms the Shannon wavelets on G.

Hence on a locally compact Abelian group, Haar and Shannon wavelets are identical. Furthermore, both the wavelets are well localized in space and frequency domain. □

4. ILLUSTRATIONS

The preceding section gives a complete theoretical discussion of construction of Haar and Shannon wavelets on a locally compact Abelian groupwith a fundamental domain. The proposed thoery is illustrated using important examples in this section.

Example 4.1. **Cantor Dyadic Group**

Let $\mathbb{F}_2 = \{0, 1\}$ be the finite field of two elements. Let $G = \mathbb{F}_2(t)$ be the group of all infinite formal sums

$$G = \{c_{n_0}t^{n_0} + c_{n_0+1}t^{n_0+1} + \cdots\},$$

where $n_0 \in \mathbb{Z}$, $c_n \in \mathbb{F}_2$. Define the addition and multiplication on G by,

$$b_n t^n + c_n t^n = (b_n + c_n)t^n,$$

$$(b_n t^n)(c_m t^m) = (b_n c_m) t^{n+m}.$$

Then the group G under addition is called the Cantor dyadic group.[5] Let $|g| = 2^{-n_0}$, where n_0 is the least integer so that $c_{n_0} \neq 0$, be the absolute value of $g \in G$.

The dual group of G is G itself and the duality pairing on G is $\chi(g_1, g_2) = e^{\pi i c_{-1}}$, where c_{-1} is the coefficient of t^{-1} in $g_1 \cdot g_2$. The group G contains a discrete subgroup Γ and a compact subgroup K,

$$\Gamma = \{c_{-n} t^{-n} + c_{-n+1} t^{-n+1} + \cdots c_{-1} t^{-1}\},$$

$$K = \{c_0 + c_1 t + c_2 t^2 + \cdots\},$$

so that $K \cap \Gamma = \{0\}$ and $K + \Gamma = G$.

Let μ be the Haar measure on G such that $\mu(K) = 1$. Here, K is the disjoint union $K = (1 + tK) \cup tK$, where $tK = \{c_1 t + c_2 t^2 + \cdots\}$. Also by translation invariance, $\mu(tK) = \mu(1 + tK)$. So $\mu(tK) = \mu(1 + tK) = \frac{1}{2}$. Similarly, $t^{-1}K = \{c_{-1} t^{-1} + c_0 + c_1 t + \cdots\}$ is a subgroup of G with measure 2 and $t^{-1}K = (t^{-1} + K) \cup K$.

We have

$$\cdots \supset t^{-2}K \supset t^{-1}K \supset K \supset tK \supset t^2 K \supset \cdots \tag{4.1}$$

and

$$\cap_{n \geq 0} t^n K = \emptyset \text{ and } \cup_{n \geq 0} t^{-n} K = G. \tag{4.2}$$

Define $A : G \to G$ by $A(g) = t^{-1} g$. Then by (4.1) and (4.2), A is an expansive automorphism with modulus $|A| = \mu(AK) = 2$ and $A^* = A$. Now

$$\begin{aligned} K^{\perp} &= \{k \in G : \chi(g,k) = 1, \forall g \in K\} \\ &= \{k \in G : c_{-1} \text{ of } g \cdot k = 0\} \\ &= K. \end{aligned} \tag{4.3}$$

Therefore, $AK/K = A^* K^{\perp}/K^{\perp}$ and both has the coset representatives $\{0, t^{-1}\}$.

Here $\widehat{G}/K^{\perp} = G/K$ and the coset representatives for G/K is the discrete group Γ. So, for any

$$k = c_{-n} t^{-n} + c_{-n+1} t^{-n+1} + \cdots c_{-1} t^{-1} + c_0 + c_1 t + c_2 t^2 + \cdots \in G,$$

$$\alpha(k) = c_{-n} t^{-n} + c_{-n+1} t^{-n+1} + \cdots c_{-1} t^{-1},$$

and

$$\beta(k) = c_0 + c_1 t + c_2 t^2 + \cdots .$$

Then for any $s \in \Gamma$, $\chi(s, \beta(k)) = \chi(s, k)$. Thus,

$$\omega_{[s]}(k) = \overline{\chi(s,k)} = \chi(s,k),$$

and $\check{\omega}_{[s]}(g) = 1_G(g-s)$. Then,

$$\tilde{\tau}_{[s]}(\widehat{f})(k) = \chi(s,k)\widehat{f}(k) \text{ and } \tau_{[s]}(f)(g) = f(g-s).$$

Define $\phi = 1_K$. Then $\widehat{\phi} = 1_{K^\perp} = 1_K$. Also, define $\widehat{\psi} = 1_{t^{-1}+K}$. Then

$$\begin{aligned}\psi(g) &= \int_G \widehat{\psi}(k)\chi(g,k)d\mu(k) \\ &= \int_{t^{-1}+K} \chi(g,k)d\mu(k) \\ &= \int_K \chi(g,t^{-1}+k)d\mu(k) \\ &= \chi(g,t^{-1})\int_K \chi(g,k)d\mu(k) \\ &= \chi(g,t^{-1})1_K \\ &= \begin{cases} 1 & ;\ g \in tK \\ -1 & ;\ g \in 1+tK \\ 0 & ;\ \text{otherwise} \end{cases}.\end{aligned} \tag{4.4}$$

Thus we have,

$$\phi(g) = \phi(t^{-1}g) + \phi(t^{-1}g - t^{-1}),$$

and

$$\psi(g) = \phi(t^{-1}g) - \phi(t^{-1}g - t^{-1}).$$

Hence, $\widehat{\phi}(k) = m_0(tk)\widehat{\phi}(tk)$ and $\widehat{\psi}(k) = m_1(tk)\widehat{\phi}(tk)$, where

$$m_0(k) = \frac{1}{2}\left(1 + \omega_{[t^{-1}]}(k)\right) = \frac{1}{2}\left(1 + \chi(t^{-1},k)\right) = \frac{1}{2}\left(1 + \chi(1,t^{-1}k)\right),$$

and

$$m_1(k) = \frac{1}{2}\left(1 - \omega_{[t^{-1}]}(k)\right) = \frac{1}{2}\left(1 - \chi(t^{-1},k)\right) = \frac{1}{2}\left(1 - \chi(1,t^{-1}k)\right).$$

This is the required Haar and Shannon wavelets on the Cantor Dyadic group.

Example 4.2. **Vilenkin Group**

Let $G = \{(y_j)\}$ be the group of all sequences,

$$y = (y_j) = (\ldots, 0, 0, y_n, y_{n+1}, y_{n+2}, \ldots),$$

where $y_j \in \{0, 1, \ldots, p-1\}$ for $j \in \mathbb{Z}$ and $y_j = 0$ for $j < n$. Define the addition operation on G by,

$$(y_j) \oplus (z_j) = y_j + z_j (\bmod p) \text{ for } j \in \mathbb{Z},$$

with $\ominus$ the inverse operation of $\oplus$. Then the group G together with the addition operation is known as Vilenkin's group.[9]

G is self-dual with character on G given by,

$$\chi(x,\xi) = \exp\left(\frac{2\pi i}{p}\sum_{j\in\mathbb{Z}} x_j\xi_{1-j}\right),\ x \in G, \xi \in G.$$

G contains a discrete subgroup Γ and a compact subgroup K given by,

$$\Gamma = \{(x_j) \in G : x_j = 0 \text{ for } j > 0\},$$

$$K = \{(x_j) \in G : x_j = 0 \text{ for } j \leq 0\}.$$

Denote the Haar measure on G by μ so that $\mu(K) = 1$. Here K is the disjoint union $K = A_0 \cup A_1 \cup \ldots \cup A_{p-1}$ where $A_j = \{(x_j) \in K : x_1 = j\}$. Then $\mu(A_j) = 1/p$.

Let $U_n = \{(x_j) \in G : x_j = 0 \text{ for } j \leq n\}$, then $K = U_0$ and

$$\cdots \supset U_{-2} \supset U_{-1} \supset U_0 \supset U_1 \supset U_2 \supset \cdots \tag{4.5}$$

and

$$\cap_{n\geq 0} U_n = \emptyset \text{ and } \cup_{n\geq 0} U_{-n} = G. \tag{4.6}$$

Define $A : G \to G$ by $A(x)_j = x_{j+1}$. Then by (4.5) and (4.6), A is an expansive automorphism with modulus $|A| = \mu(AK) = \mu(U_{-1}) = p$ and $A^* = A$. Now

$$\begin{aligned} K^\perp &= \{\xi \in G : \chi(x,\xi) = 1,\ \forall (x_j) \in K\} \\ &= \{\xi \in G : \exp\left(\frac{2\pi i}{p}\sum_{j\in\mathbb{Z}} x_j\xi_{1-j}\right) = 1,\ \forall (x_j) \in K\} \\ &= \{\xi \in G : \sum_{j=1}^{\infty} x_j\xi_{1-j} = 0,\ \forall x_j = 0, 1, \ldots p-1\} \\ &= \{\xi \in G : \xi_j = 0 \text{ for } j \leq 0\} \\ &= K. \end{aligned} \tag{4.7}$$

Therefore, $AK/K = A^*K^\perp/K^\perp$ and both has the coset representatives $\{\sigma_0, \sigma_1, \ldots \sigma_{p-1}\}$, where $(\sigma_j)_l = \begin{cases} j & ;\ l = 0 \\ o & ;\ \text{otherwise} \end{cases}$.

Here $\widehat{G}/K^\perp = G/K$ and the coset representatives for G/K is the discrete group Γ. So, for any

$$\xi = (\xi_j) = (\ldots, 0, x_{-k}, \ldots, x_1, x_0, x_1, x_2, \ldots) \in G,$$

$$\alpha(\xi) = (\ldots, 0, x_{-k}, \ldots, x_1, x_0, 0, \ldots),$$

and

$$\beta(\xi) = (\ldots, 0, x_1, x_2, \ldots).$$

Then for any $s \in \Gamma$, $\chi(s, \beta(\xi)) = \chi(s, \xi)$. Thus,

$$\omega_{[s]}(\xi) = \overline{\chi(s, \xi)},$$

and $\check{\omega}_{[s]}(x) = 1_G(x - s)$. Then,

$$\tau_{[s]}(f)(x) = f(x - s).$$

Define $\phi = 1_K$. Then $\widehat{\phi} = 1_{K^\perp} = 1_K$. Also, for $j = 0, 1, \ldots p-1$ define $\widehat{\psi_j} = 1_{\sigma_j + K}$. Then

$$\begin{aligned} \psi_j(x) &= \int_G \widehat{\psi_j}(\xi)\chi(x, \xi)d\mu(\xi) \\ &= \int_{\sigma_j + K} \chi(x, \xi)d\mu(\xi) \\ &= \chi(x, \sigma_j)1_K \\ &= \begin{cases} \exp\left(\frac{2\pi i j l}{p}\right) & ; x \in A_l \\ 0 & ; x \notin K \end{cases}. \end{aligned} \tag{4.8}$$

Thus we have,

$$\phi(x) = \phi(Ax) + \phi(Ax - \sigma_1) + \cdots + \phi(Ax - \sigma_{p-1}),$$

and

$$\psi_j(x) = \phi(Ax) + e^{2\pi i j/p}\phi(Ax - \sigma_1) + \cdots + e^{2\pi i j(p-1)/p}\phi(Ax - \sigma_{p-1}).$$

Hence, $\widehat{\phi}(\xi) = m_0(A^{-1}\xi)\widehat{\phi}(A^{-1}\xi)$ and $\widehat{\psi_j}(\xi) = m_j(A^{-1}\xi)\widehat{\phi}(A^{-1}\xi)$, where

$$m_0(\xi) = \frac{1}{p}\left(1 + \sum_{l=1}^{p-1} \omega_{[\sigma_l]}(\xi)\right),$$

and

$$m_j(\xi) = \frac{1}{p}\left(1 + \sum_{l=1}^{p-1} e^{2\pi i j l/p}\omega_{[\sigma_l]}(\xi)\right).$$

Here $\omega_{[\sigma_l]}(\xi) = e^{-2\pi i l \xi_1/p}$, where $\xi = (\xi_j)$. This is the required Haar and Shannon wavelets on the Vilenkin's group.

Example 4.3. **p-adic field**

Consider the completion field of $\mathbb{Q}$ with respect to the norm $|\cdot|_p$ defined by,

$$|x|_p = \begin{cases} 0 & ; x = 0 \\ p^{-\gamma} & ; x \neq 0, x = p^{\gamma}\frac{m}{n}, \end{cases}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $m, n \in \mathbb{Z}$ not divisible by p. Denote the above field as $G = \mathbb{Q}_p$.[14] The canonical form of $x \in \mathbb{Q}_p$, $x \neq 0$ is,

$$x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + \cdots),$$

where $\gamma \in \mathbb{Z}$, $x_j \in \{0, 1, \ldots p-1\}$, $x_0 \neq 0$. Then for this $x \in \mathbb{Q}_p$, the fractional part of x is,

$$\{x\}_p = \begin{cases} 0 & ; \gamma(x) \geq 0 \text{ or } x = 0 \\ p^{\gamma}(x_0 + x_1 p + \cdots + x_{-\gamma-1}p^{-\gamma-1}) & ; \gamma(x) < 0 \end{cases}.$$

The dual group of G is G itself and the character on G is defined as,

$$\chi(x, \xi) = e^{2\pi i \{x\xi\}_p},$$

where $\{\cdot\}_p$ is the fractional part of a number. Here G contains a compact subgroup K,

$$K = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Let μ be the Haar measure on G with $\mu(K) = 1$. For any $x \in K$ with $x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + \cdots)$, $|x|_p \leq 1 \Rightarrow \gamma \geq 0$. So we can write,

$$K = \{x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + \cdots) \in \mathbb{Q}_p : \gamma \geq 0\}.$$

Put $A_0 = \{x \in K : \gamma > 0\}$, $A_j = \{x \in K : \gamma = 0 \text{ and } x_0 = j\}$, for $j = 1, 2, \ldots p-1$. Then $K = A_0 \cup A_1 \cup \ldots \cup A_{p-1}$.

Denote $B_{\gamma}(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^{\gamma}\}$. Then $K = B_0(0)$ and

$$\cdots \supset B_2(0) \supset B_1(0) \supset B_0(0) \supset B_{-1}(0) \supset B_{-2}(0) \supset \cdots \tag{4.9}$$

and

$$\cap_{n \leq 0} B_n(0) = \emptyset \text{ and } \cup_{n \geq 0} B_n(0) = G. \tag{4.10}$$

Define $A : G \to G$ by $A(x) = \frac{1}{p}x$. Then by (4.9) and (4.10), A is an expansive automorphism with modulus $|A| = \mu(AK) = \mu(B_1(0)) = p$ and $A^* = A$. Now

$$\begin{aligned} K^{\perp} &= \{\xi \in G : \chi(x, \xi) = 1, \forall x \in K\} \\ &= \{\xi \in G : e^{2\pi i \{x\xi\}_p} = 1, \forall x \in K\} \\ &= \{\xi \in G : \{x\xi\}_p = 0 \, \forall x \in K\} \\ &= \{\xi = p^{\lambda}(\xi_0 + \xi_1 p + \cdots) \in G : \lambda \geq 0\} \\ &= K. \end{aligned} \tag{4.11}$$

Therefore, $AK/K = A^*K^{\perp}/K^{\perp}$ and both has the coset representatives $\{\sigma_0, \sigma_1, \dots \sigma_{p-1}\}$, where $\sigma_j = \frac{j}{p}$.

Here $\widehat{G}/K^{\perp} = G/K \approx I_p$ where

$$I_p = \{a = p^{-\gamma}(a_0 + a_1 p + \cdots + a_{\gamma-1}p^{\gamma-1}) : \gamma \in \mathbb{N},\ a_j \in \{0, 1, \dots p-1\}\}.$$

So, for any

$$\xi = p^{\lambda}(\xi_0 + \xi_1 p + \cdots) \in G,$$
$$\alpha(\xi) = \{\xi\}_p,$$

and

$$\beta(\xi) = \begin{cases} \xi & ; \lambda \geq 0 \text{ or } x = 0 \\ \xi_{-\lambda} + \xi_{-\lambda+1}p + \cdots & ; \lambda < 0 \end{cases}.$$

Then for any $a \in I_p$, $\omega_{[a]}(\xi) = \overline{\chi(a, \beta(\xi))}$ and $\tau_{[a]}(f)(x) = (\omega_{[a]} * f)(x)$.

Define $\phi = 1_K$. Then $\widehat{\phi} = 1_{K^{\perp}} = 1_K$. Also, for $j = 0, 1, \dots p-1$ define $\widehat{\psi_j} = 1_{\sigma_j + K}$. Then

$$\begin{aligned} \psi_j(x) &= \int_G \widehat{\psi_j}(\xi)\chi(x, \xi)d\mu(\xi) \\ &= \int_{\sigma_j + K} \chi(x, \xi)d\mu(\xi) \\ &= \chi(x, \sigma_j)1_K \\ &= \begin{cases} 1 & ; x \in A_0 \\ \exp\left(\frac{2\pi i j l}{p}\right) & ; x \in A_l, l = 1, 2, \dots p-1 \\ 0 & ; x \notin K \end{cases}. \end{aligned} \tag{4.12}$$

Thus we have,

$$\phi(x) = \phi(Ax) + \phi(Ax - \sigma_1) + \cdots + \phi(Ax - \sigma_{p-1}),$$

and

$$\psi_j(x) = \phi(Ax) + e^{2\pi i j/p}\phi(Ax - \sigma_1) + \cdots + e^{2\pi i j(p-1)/p}\phi(Ax - \sigma_{p-1}).$$

Hence, $\widehat{\phi}(\xi) = m_0(A^{-1}\xi)\widehat{\phi}(A^{-1}\xi)$ and $\widehat{\psi_j}(\xi) = m_j(A^{-1}\xi)\widehat{\phi}(A^{-1}\xi)$, where

$$m_0(\xi) = \frac{1}{p}\left(1 + \sum_{l=1}^{p-1} \omega_{[\sigma_l]}(\xi)\right),$$

and

$$m_j(\xi) = \frac{1}{p}\left(1 + \sum_{l=1}^{p-1} e^{2\pi i j l/p}\omega_{[\sigma_l]}(\xi)\right).$$

This is the required Haar and Shannon wavelets on the p-adic field.

5. CONCLUSION

A novel method with sufficient theoretical background is proposed to construct a scaling function ϕ and a multiresolution analysis for $L^2(G)$, where G is a locally compact Abelian group having a fundamental domain. Further, using the developed scaling function and multiresolution analysis, the authors constructed the corresponding wavelet system and proved that the Haar and Shannon wavelets are same in the present context. Finally the developed method is illustrated through examples.

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