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Taylor Wavelet Based Numerical Method for Hyperbolic Inverse Problem with a Control Parameter

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Abstract

In this article, we develop a highly efficient and accurate wavelet collocation method based on Taylor wavelet for the identification of control parameter in the hyperbolic partial differential equations (PDEs). In the proposed method, highest order derivative is represented in terms of Taylor wavelets and required unknown terms of the partial differential equation is obtained using successive integration. Taylor series approximation is used in order to identify the control parameter. Error analysis is carried out in order to ensure the convergence of the method. Numerical results validate the proposed collocation method based on Taylor wavelets.

Keywords: Taylor wavelet, Control parameter, Hyperbolic inverse PDEs, Collocation method, Taylor series

1. INTRODUCTION

In the last few years, there has been significant interest in the develoment of accuarte and efficient methods for the numerical solution of hyperbolic inverse problem[7] with control parameter. These problems have many applications in various fields like acoustics, seismology, fluid dynamics, electrical circuits, biological systems like HIV infection models.

In this article, we develop a numerical method based on Taylor wavelet for one dimensional hyperbolic inverse problem with a source control parameter given by

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + X(t)u(x,t) + \phi(x,t),\ \ x \in (0,1),\ \ 0 \le t \le T, \tag{1.1}$$

with initial condition

$$u(x,0) = f_1(x), \qquad \frac{\partial u}{\partial t}(x,0) = f_2(x),\ x \in (0,1), \tag{1.2}$$

and Dirichlet boundary conditions

$$u(0,t) = g_1(t), \qquad u(1,t) = g_2(t), \quad 0 \leq t \leq T, \tag{1.3}$$

subject to the overspecified condition

$$u(x^*,t) = E(t), \quad 0 \leq t \leq T, \tag{1.4}$$

where x^*, is a fixed point such that $0 < x^* < 1$ and ϕ, f_1, f_2, g_1, g_2 and E are known functions. Let us assumed that $E(t) \neq 0$. We have to determine unknown functions $u(x,t)$ and $X(t)$ simultaneously, where $u(x,t)$ stands for the solution of the given problem and $X(t)$ is the source control parameter.

Parameter identification for the hyperbolic inverse problem by Haar wavelets has been discussed in[7] which based on wavelet collocation method.However, the Haar wavelet is not continuous and hence it is not differentiable. this can limit its use in certain applications. The Haar wavelet have compact support, they are non-zero only over a finite interval.In this work we solve the hyperbolic inverse problem by Taylor wavelet method. Taylor wavelet method has property to approximate an unknown function, by using a series expansion of taylor wavelets. It belongs to the broader field of wavelet based numerical technique which has the properties of wavelets to localized in both time and frequencey domain. Taylor wavelet has computational efficiency, accuracy, versatility, compact support and flexibility which make it interesting for researchers. A numerical method based on Harmonic wavelet has been proposed by Cattani[1] for the solution of nonlinear PDE. Haar wavelet method for solving lumped and distributed parameter are studied by Chen *et al.*[2] Gopal *et al.*[11] developed the Chebyshev and Taylor wavelet method for parabolic inverse problem.

Taylor wavelet method(TWM) gives a powerful tool for the solution of hyperbolic PDEs. It gives a high accuracy and computational efficiency, which make it valuable for a wide range of application. In this technique we use a series expansion of Taylor wavelets. Any function can approximated by using a series expansion of Taylor wavelets. As the terms of the series, the approximation converges uniformly.

From the above introduce significant studies, the main objective of this study is to developed a comparative work based on TWM for hyperbolic inverse PDEs. We have evaluate the unknown parameter by using Taylor series approximation. The error estimate, is ensure the convergence of the proposed method.

In this article, the content has been organized as follows: In Section 2, some theoretical basic background of Taylor wavelet is reviewed. In Section 3, Taylor wavelet collocation method is implemented for the parameter identification in one dimensional hyperbolic inverse problem. In section 4, Convergence analysis for the given method is carried out. In section 5, numerical results are discussed. In section 6, a brief conclusion and future work has been discussed.

2. BASIC BACKGROUND

In this section, we will provide a brief introduction on the Taylor wavelets and their first and second integral forms of these wavelets. We point to our some fundamental references for Taylor wavelets, (see[11–14])

Before presenting the some basic introduction we clarify the notations that have been used. Let $J_{nm}(x)$ represents the wavelet method for Taylor wavelets . We denote the first and second integral forms for both methods by $K_{nm}(x)$ and $L_{nm}(x)$, respectively.

Let $n = 1, 2, \ldots, 2^{k-1}$ for $k \in \mathbb{Z}^+$ and $m = 0, 1, \ldots, M-1$ where M stands for the degree of the polynomial, the Taylor wavelets definied as

$$J_{nm}(x) = \begin{cases} 2^{\frac{k-1}{2}} \hat{T}_m \left(2^{k-1}x - n + 1\right), & \text{if } \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}; \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

where $\hat{T}_m\,(x) = \sqrt{2m+1}x^m$ for $m = 0, 1, 2, \ldots, M-1$. Here degree of polynomial represented by x^m of degree m which forms an orthonormal basis over the interval [0,1].

From the above some basic definitions, any function $f(x) \in L^2_\alpha[0,1]$ can be expressed in terms of the Taylor wavelet as follows;

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} h_{nm} J_{nm}(x), \tag{2.2}$$

where h_{nm} represent the wavelet coefficients for the Taylor wavelet, and they are constituted by

$$h_{nm} = \langle f(x), J_{nm}(x)\rangle = \int_0^1 f(x)J_{nm}(x)dx, \text{for the Taylor wavelet;} \tag{2.3}$$

Moreover, the approximation of $f(x)$ is given by

$$f(x) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} h_{nm} J_{nm}(x) = \boldsymbol{H}\boldsymbol{J}(x), \tag{2.4}$$

where $\boldsymbol{H}$ is a $1 \times 2^{k-1}M$ vector given by

$$\boldsymbol{H} = \left[h_{10},\, h_{11}, \ldots, h_{1(M-1)},\, h_{20},\, h_{21}, \ldots,\, h_{2(M-1)}, \ldots, h_{2^{k-1}0},\, h_{2^{k-1}1}, \ldots, h_{2^{k-1}(M-1)}\right], \tag{2.5}$$

and

$$\boldsymbol{J}(x) = \left[J_{10}(x)\dots J_{M-1}(x), J_{20}(x)\dots J_{2(M-1)}(x),\dots, J_{2^{k-1}0}(x)\dots J_{2^{k-1}(M-1)}(x)\right]^T. \quad (2.6)$$

Note that, in Eq.(2.6) replacing the suitable collocation points $\boldsymbol{J}(x)$ returns $2^{k-1}M \times 2^{k-1}M$ matrix.

We introduce this introductory section with the required integral forms of the Taylor wavelet.The first and second integral forms of the Taylor wavelet are

$$K_{nm}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{n-1}{2^{k-1}}; \\ \frac{2^{\left(m+\frac{1}{2}\right)(k-1)} m!\sqrt{2m+1}}{(m+1)!}\left(x - \frac{n-1}{2^{k-1}}\right)^{m+1}, & \text{if } \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}; \\ \frac{2^{\left(m+\frac{1}{2}\right)(k-1)} m!\sqrt{2m+1}}{(m+1)!}\left(x - \frac{n-1}{2^{k-1}}\right)^{m+1} - P_1(x), & \text{if } \frac{n}{2^{k-1}} \le x \le 1, \end{cases} \quad (2.7)$$

and

$$L_{nm}(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{n-1}{2^{k-1}}; \\ \frac{2^{\left(m+\frac{1}{2}\right)(k-1)} m!\sqrt{2m+1}}{(m+1)!}\left(x - \frac{n-1}{2^{k-1}}\right)^{m+1}, & \text{if } \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}; \\ \frac{2^{\left(m+\frac{1}{2}\right)(k-1)} m!\sqrt{2m+1}}{(m+1)!}\left(x - \frac{n-1}{2^{k-1}}\right)^{m+1} - P_2(x), & \text{if } \frac{n}{2^{k-1}} \le x \le 1, \end{cases} \quad (2.8)$$

where

$$P_i(x) = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \frac{2^{\left(j+\frac{1}{2}\right)(k-1)} j!\sqrt{2m+1}}{(j+i)!}\left(x - \frac{n}{2^{k-1}}\right)^{j+i}, \quad i = 1,2. \quad (2.9)$$

For the simplicity and understandability, we avoid the unnecessary notations. Thus, for any continuous $f(x)$ in this section, we have the following relations:

$$\begin{aligned} f(x) &\simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} h_{nm}J_{nm}(x) = \boldsymbol{H}\boldsymbol{J}(x), \\ \int_0^x f(\tau)h\tau &\simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} h_{nm}K_{nm}(x) = \boldsymbol{H}\boldsymbol{K}(x), \\ \int_0^x\int_0^\xi f(\tau)h\tau\, d\xi &\simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} h_{nm}L_{nm}(x) = \boldsymbol{H}\boldsymbol{L}(x). \end{aligned}$$

Before ending the section, a further introduction, which is necessary for constructing the numerical scheme, is needed to give.For that purpose, we finalized this section by defining the Kronecker product, that is denoted by $\otimes$ notation.The Kronnecker product of two vectors results in a matrix where each element is the product of corresponding elements from the two vectors. More precisely, for any $\boldsymbol{x} = \left[x_1,\, x_2, \ldots, x_{2^{k-1}(M-1)}\right]$ we have

$$\boldsymbol{x} \otimes \boldsymbol{L}(1) = \begin{bmatrix} x_1 L_{10}(1) & \cdots & x_1 L_{20}(1) & \cdots & x_1 L_{2^{k-1}(M-1)}(1) \\ x_2 L_{10}(1) & \cdots & x_2 L_{20}(1) & \cdots & x_2 L_{2^{k-1}(M-1)}(1) \\ \vdots & \ddots & \cdots & \ddots & \vdots \\ x_{2^{k-1}(M-1)-1} L_{10}(1) & \cdots & x_{2^{k-1}(M-1)-1} L_{20}(1) & \cdots & x_{2^{k-1}(M-1)-1} L_{2^{k-1}(M-1)}(1) \\ x_{2^{k-1}(M-1)} L_{10}(1) & \cdots & x_{2^{k-1}(M-1)} L_{20}(1) & \cdots & x_{2^{k-1}(M-1)} L_{2^{k-1}(M-1)}(1) \end{bmatrix}, \tag{2.10}$$

Where

$$\boldsymbol{L}(1) = \left[L_{10}(1), \ldots, L_{1(M-1)}(1), L_{20}(1), \ldots, L_{2(M-1)}(1), \ldots, L_{2^{k-1}0}(1), \ldots, L_{2^{k-1}(M-1)}(1)\right]. \tag{2.11}$$

3. IMPLEMENTATION OF TAYLOR WAVELET METHOD FOR HYPERBOLIC INVERSE PROBLEM

In this section, we develop a numerical method based on Taylor wavelets to solve hyperbolic inverse partial differential equation with a source control parameter given by (1.1- 1.4).

Let N_s denote the number of divisions of the time interval such that $\Delta t = \frac{T}{N_s}$. On $t \in [t_s, t_{s+1})$. Notice that $u(x,t)$ and $U(x,t)$ represent the numerical solution and exact solution, respectively. Let us assume that

$$\left(\frac{\partial^4 u}{\partial x^2 \partial t^2}\right)(x,t) = \boldsymbol{H}\boldsymbol{J}(x), \qquad t \in [t_s, t_{s+1}] \tag{3.1}$$

Integrating Eq.(3.1) with respect to t from t_s to t, we get

$$\left(\frac{\partial^3 u}{\partial x^2 \partial t}\right)(x,t) = (t-t_s)\boldsymbol{H}\boldsymbol{J}(x) + \left(\frac{\partial^3 u}{\partial x^2 \partial t}\right)(x,t_s) \tag{3.2}$$

Integrating Eq. (3.2) with respect to t from t_s to t, we get

$$\left(\frac{\partial^2 u}{\partial x^2}\right)(x,t) = \frac{(t-t_s)^2}{2}\boldsymbol{H}\boldsymbol{J}(x) + (t-t_s)\left(\frac{\partial^3 u}{\partial x^2 \partial t}\right)(x,t_s) + \left(\frac{\partial^2 u}{\partial x^2}\right)(x,t_s) \tag{3.3}$$

Equation (3.3) can be written as

$$\left(\frac{\partial^2 u}{\partial x^2}\right)(x,t) = \frac{(t-t_s)^2}{2}\boldsymbol{H}\boldsymbol{J}(x) + (t-t_s)\left(\frac{\partial^2}{\partial x^2}\frac{\partial u}{\partial t}\right)(x,t_s) + \left(\frac{\partial^2 u}{\partial x^2}\right)(x,t_s) \tag{3.4}$$

Putting the initial condition in Eq.(3.4), we have

$$\left(\frac{\partial^2 u}{\partial x^2}\right)(x,t) = \frac{(t-t_s)^2}{2}\boldsymbol{H}\boldsymbol{J}(x) + (t-t_s)f_2''(x) + f_1''(x) \tag{3.5}$$

Integrating Eq.(3.1) with respect to x from 0 to x, we have

$$\left(\frac{\partial^3 u}{\partial x \partial t^2}\right)(x,t) = \boldsymbol{H}\boldsymbol{K}(x) + \left(\frac{\partial^3 u}{\partial t^2 \partial x}\right)(0,t) \tag{3.6}$$

Integrating Eq.(3.6) with respect to x from 0 to x, we have

$$\left(\frac{\partial^2 u}{\partial t^2}\right)(x,t) = \boldsymbol{H}\boldsymbol{L}(x) + x\left(\frac{\partial^3 u}{\partial t^2 \partial x}\right)(0,t) + \left(\frac{\partial^2 u}{\partial t^2}\right)(0,t) \tag{3.7}$$

Putting $x = 1$ in Eq.(3.7) and applying boundary condition

$$\left(\frac{\partial^2 g_2}{\partial t^2}\right)(t) - \boldsymbol{H}\boldsymbol{L}(1) - \left(\frac{\partial^2 g_1}{\partial t^2}\right)(t) = \left(\frac{\partial^3 u}{\partial t^2 \partial x}\right)(0,t) \tag{3.8}$$

From Eq.(3.7) and Eq.(3.8), we have

$$\left(\frac{\partial^2 u}{\partial t^2}\right)(x,t) = \boldsymbol{H}\Big(\boldsymbol{L}(x) - x \otimes \boldsymbol{L}(1)\Big) + x\left(\frac{\partial^2 g_2}{\partial t^2}(t) - \frac{\partial^2 g_1}{\partial t^2}(t)\right) + \left(\frac{\partial^2 g_1}{\partial t^2}\right)(t) \tag{3.9}$$

Integrate Eq.(3.7) with respect to t from t_s to t, we have

$$\begin{aligned}\left(\frac{\partial u}{\partial t}\right)(x,t) = {}& (t-t_s)\boldsymbol{H}\boldsymbol{L}(x) \\ & + x\left(\frac{\partial^2 u}{\partial x \partial t}(0,t) - \frac{\partial^2 u}{\partial x \partial t}(0,t_s)\right) + \left(\frac{\partial u}{\partial t}\right)(0,t) - \left(\frac{\partial u}{\partial t}\right)(0,t_s) \\ & + \left(\frac{\partial u}{\partial t}\right)(x,t_s)\end{aligned} \tag{3.10}$$

Putting $x = 1$ in Eq.(3.10) and apply boundary condition, we have

$$\begin{aligned}&\left(\frac{\partial^2 u}{\partial x \partial t}(0,t) - \frac{\partial^2 u}{\partial x \partial t}(0,t_s)\right) \\ &= \left(\frac{\partial g_2}{\partial t}(t) - (t-t_s)\boldsymbol{H}\boldsymbol{L}(1) - \frac{\partial g_1}{\partial t}(t) + \frac{\partial g_1}{\partial t}(t_s) - \frac{\partial g_2}{\partial t}(t_s)\right)\end{aligned} \tag{3.11}$$

From Eq.(3.10) and Eq.(3.11), we have

$$\begin{aligned}\left(\frac{\partial u}{\partial t}\right)(x,t) &= (t-t_s)\boldsymbol{H}\Big(\boldsymbol{L}(x) - x \otimes \boldsymbol{L}(1)\Big) + x\Big(g_2'(t) - g_1'(t) + g_1'(t_s) - g_2'(t_s)\Big) \\ &\quad + \left(\frac{\partial u}{\partial t}\right)(0,t) - \left(\frac{\partial u}{\partial t}\right)(0,t_s) + \left(\frac{\partial u}{\partial t}\right)(x,t_s)\end{aligned} \tag{3.12}$$

Integrating Eq.(3.10) with respect to t from t_s to t, we have

$$\begin{aligned}u(x,t) &= \frac{(t-t_s)^2}{2}\boldsymbol{H}\boldsymbol{L}(x) + x\left(\frac{\partial u}{\partial x}(0,t) - \frac{\partial u}{\partial x}(0,t_s) - (t-t_s)\frac{\partial^2 u}{\partial x \partial t}(0,t_s)\right) + u(0,t) \\ &\quad - u(0,t_s) - (t-t_s)\left(\frac{\partial u}{\partial t}\right)(0,t_s) + (t-t_s)\left(\frac{\partial u}{\partial t}\right)(x,t_s) + u(x,t_s)\end{aligned} \tag{3.13}$$

Putting $x = 1$ in Eq.(3.13) and apply boundary condition, we have

$$\left(\frac{\partial u}{\partial x}(0,t) - \frac{\partial u}{\partial x}(0,t_s) - (t-t_s)\frac{\partial^2 u}{\partial x \partial t}(0,t_s)\right) = g_2(t) - \frac{(t-t_s)^2}{2}\boldsymbol{H}\boldsymbol{L}(1)$$
$$- g_1(t) + g_1(t_s) + (t-t_s)g_1'(t) - (t-t_s)g_2'(t_s) - g_2(t_s)$$

From Eq.(3.13) and Eq.(3.14), we have

$$\begin{aligned} u(x,t) &= \frac{(t-t_s)^2}{2}\boldsymbol{H}\left(\boldsymbol{L}(x) - x \otimes \boldsymbol{L}(1)\right) + \\ & x\left(g_2(t) - g_1(t) + g_1(t_s) + (t-t_s)g_1'(t) - (t-t_s)g_2'(t_s) - g_2(t_s)\right) + u(0,t) \\ & -u(0,t_s) - (t-t_s)\left(\frac{\partial u}{\partial t}\right)(0,t_s) + (t-t_s)\left(\frac{\partial u}{\partial t}\right)(x,t_s) + u(x,t_s) \end{aligned} \tag{3.14}$$

We discretizing Eq.(1) by assuming $t \to t_{s+1}$ and $x \to x_l$ we have

$$\left(\frac{\partial^2 u}{\partial t^2}\right)(x_l,t_{s+1}) - \left(\frac{\partial^2 u}{\partial x^2}\right)(x_l,t_{s+1}) - X(t_{S+1})u(x_l,t_{s+1}) = \phi(x_l,t_{s+1}) \tag{3.15}$$

Substituting the expressions for $\left(\frac{\partial^2 u}{\partial t^2}\right)(x_l,t_{s+1}), \left(\frac{\partial^2 u}{\partial x^2}\right)(x_l,t_{s+1}), u(x_l,t_{s+1})$ in Eq.(3.16) we get the following

$$\begin{aligned} &\boldsymbol{H}\left(\boldsymbol{L}(x) - x \otimes \boldsymbol{L}(1)\right) + x_l\left(\frac{\partial^2 g_2}{\partial t^2}(t_{s+1}) - \frac{\partial^2 g_1}{\partial t^2}(t_{s+1})\right) + \left(\frac{\partial^2 g_1}{\partial t^2}\right)(t_{s+1}) \\ &- \left[\frac{(t_{s+1}-t_s)^2}{2}\boldsymbol{H}\boldsymbol{J}(x) + (t_{s+1}-t_s)f_2''(x) + f_1''(x)\right] \\ &- X(t_{s+1})[\frac{(t_{s+1}-t_S)^2}{2}\boldsymbol{H}\left(\boldsymbol{L}(x) - x \otimes \boldsymbol{L}(1)\right) && (3.16) \\ &+ x_l\left(g_2(t_{s+1}) - g_1(t_{s+1}) + g_1(t_s) + (t_{s+1}-t_s)g_1'(t_{s+1}) - (t_{s+1}-t_s)g_2'(t_s) - g_2(t_s)\right) \\ &+ u(0,t_{s+1}) - u(0,t_s) - (t_{s+1}-t_s)\frac{\partial u}{\partial t}(0,t_s) + (t_{s+1}-t_s)\frac{\partial u}{\partial t}(x_l,t_s) && (3.17) \\ &+ u(x_l,t_s)] = \phi(x_l,t_{s+1}) && (3.18) \end{aligned}$$

Which can be reduced into a matrix equation such that

$$\boldsymbol{M}\boldsymbol{H} = \boldsymbol{f} \tag{3.19}$$

where

$$\begin{aligned}\boldsymbol{M} =&(\boldsymbol{L}(x) - x \otimes \boldsymbol{L}(1)) - \left(\frac{(t_{s+1} - t_s)^2}{2}\right)\boldsymbol{H}\boldsymbol{J}(x)\\ &- X(t_{s+1})\frac{(t_{s+1} - t_s)^2}{2}(\boldsymbol{L}(x) - x \otimes \boldsymbol{L}(1))\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}&\boldsymbol{f} = \phi(x_l, t_{s+1}) - x_l\left(\frac{\partial^2 g_2}{\partial t^2}(t_{s+1}) - \frac{\partial^2 g_1}{\partial t^2}(t_{s+1})\right) - \left(\frac{\partial^2 g_1}{\partial t^2}\right)(t_{s+1})\\ &+ \left((t_{s+1} - t_s)f_2''(x) + f_1''(x)\right)\\ &+ X(t_{s+1})\Bigg(u(0, t_{s+1}) - u(0, t_s) - (t_{s+1} - t_s)\frac{\partial u}{\partial t}(0, t_s) + (t_{s+1} - t_s)\frac{\partial u}{\partial t}(x_l, t_s)\\ &+ u(x_l, t_s) + x_l(g_2(t_{s+1}) - g_1(t_{s+1}) + g_1(t_s) + (t_{s+1}\\ &- t_s)g_1'(t_{s+1}) - (t_{s+1} - t_s)g_2'(t_s) - g_2(t_s))\Bigg)\end{aligned} \tag{3.21}$$

Using matrix equation (3.18), we obtain the wavelet coefficients at $t = t_{s+1}$ for both the Taylor wavelet via the gmres package in MATLAB.

3.1. Numerical approximation of the parameter

From Equation (1), we have

$$\frac{\partial^2 u}{\partial t^2}(x^*, t) = \frac{\partial^2 u}{\partial x^2}(x^*, t) + X(t)u(x^*, t) + \phi(x^*, t) \tag{3.22}$$

Using the given overspecification condition (1.4), we have

$$\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}(x^*, t) + X(t)u(x^*, t) + \phi(x^*, t) \tag{3.23}$$

This implies

$$X(t) = \frac{\frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}(x^*, t) - \phi(x^*, t)}{E(t)} \tag{3.24}$$

The numerical solution for the source control parameter evaluated by the Taylor expansion, at $t = t_{s+1}$ as follows:

$$X(t_{s+1}) = \frac{\frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}(x^*, t_s) - \frac{\partial^3 u}{\partial t \partial x^2}(x^*, t_s) + \phi(x^*, t_s)}{E(t)} + \mathcal{O}(\Delta t^2) \tag{3.25}$$

4. CONVERGENCE ANALYSIS

4.1. The error estimations for the TWM

Theorem 4.1. *Let* $f(x) \in \mathcal{C}^2[0,1]$ *is bounded second order derivative that is,* $|f''(x)| \leq R$. $f(x)$ *may be express as an infinite sum of Taylor wavelets. Moreover, this series converges uniformly to* $f(x)$. *that is*

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} h_{nm} J_{nm}(x)$$

Notice that $h_{nm} = \langle f(x), J_{nm}(x) \rangle$ *where* $\langle \cdot, \cdot \rangle$ *denotes the inner product in* $L^2[0,1]$ *given in Equation (2.3) and* $J_{nm}(x)$ *are the Taylor wavelets,*

$$|h_{nm}| \leq \frac{R\sqrt{2m+1}}{n^{\frac{5}{2}}(m+1)(m+2)(m+3)}, \quad m > 1.$$

notice that n *denotes the resolutions of the interval then* $n = 1,\, 2,\, \ldots, 2^{k-1}$ *for* $k \in \mathbb{Z}^+$ *and* m *denotes the degree of polynomials.*

Proof. From the definition of h_{nm},we have

$$\begin{aligned} h_{nm} &= \int_0^1 f(x) J_{nm}(x) dx, \\ &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(x) 2^{\frac{k-1}{2}} \hat{T}_m(2^{k-1}x - n + 1) dx. \end{aligned} \tag{4.1}$$

Changing the variables $\tau = 2^k x - n + 1$ leads the

$$h_{nm} = \int_0^1 f\left(\frac{\tau + n - 1}{2^{k-1}}\right) 2^{-\frac{k-1}{2}} \sqrt{2m+1} (\tau)^m d\tau.$$

With the process of twice integration by parts, we get

$$|h_{nm}| \leq \frac{\sqrt{2m+1}}{2^{5\frac{k-1}{2}}(m+1)(m+2)} \left| \int_0^1 f''\left(\frac{\tau+n-1}{2^{k-1}}\right) (\tau)^{m+2} d\tau \right|.$$

Now we can use the Cauchy-Schwartz inequality

$$\begin{aligned} |h_{nm}| &\leq \frac{\sqrt{2m+1}}{2^{5\frac{k-1}{2}}(m+1)(m+2)} \left| \int_0^1 f''\left(\frac{\tau+n-1}{2^{k-1}}\right) d\tau \right| \left| \int_0^1 (\tau)^{m+2} d\tau \right|, \\ &\leq \frac{R\sqrt{2m+1}}{n^{\frac{5}{2}}(m+1)(m+2)(m+3)}. \end{aligned} \tag{4.2}$$

As a outcome of Equation (4.2), $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} h_{nm}$ is absolutely convergent as $k, M \to \infty$. This confirm the uniform convergence of $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} h_{nm} J_{nm}(x)$ to the function $f(x).\lim_{k,M\to\infty}\left|f(x)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}h_{nm}J_{nm}(x)\right|\to 0$.Next proofs also confirm this convergence result.

□

From Theorem (4.1.1),Theorem (4.1.2) also describes the accuracy of the TWM and its integral forms. Let γ_m denotes the accuraccy of the TWM.

Theorem 4.2. *Let $f(x)\in\mathcal{C}^2[0,1]$ is bounded second order derivative that is, $|f''(x)|\leq R$. Then, we have the following error analysis for both the TWM and its integral forms*

$$\gamma_{nm}\leq R\left(\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\frac{2m+1}{n^5(m+1)^2(m+2)^2+(m+3)^2}\right)^{1/2}, \tag{4.3}$$

where n denotes the resolutions of the interval for which $n=1,\,2,\,\ldots,2^{k-1}$ for $k\in\mathbb{Z}^+$ and m denotes the degree of polynomials.

Proof. For the accuracy of the TWM and its integral forms, we use the following method as follows

$$\begin{aligned}\left\|f(x)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}h_{nm}J_{nm}(x)\right\|^2 &= \int_0^1\left|\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}h_{nm}J_{nm}(x)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M_1}h_{nm}J_{nm}\right|^2 dx\\ \gamma_{nm}^2 &\leq \int_0^1\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|h_{nm}|^2 J_{nm}^2(x)dx. \qquad (4.4)\end{aligned}$$

Now we use the triangle inequality implies that

$$\begin{aligned}\gamma_{nm}^2 &\leq \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|h_{nm}|^2\int_0^1 J^2nm(x)dx,\\ &\leq \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|h_{nm}|^2\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left(2^{\frac{k-1}{2}}\sqrt{2m+1}(2^{k-1}x-n+1)^m\right)^2 dx, \qquad (4.5)\\ &\leq \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|h_{nm}|^2 \qquad (4.6)\end{aligned}$$

From Equation (4.2), the accuracy of estimation of the TWM can be defined as follows

$$\gamma_{nm}^2\leq R\left(\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\frac{2m+1}{n^5(m+1)^2(m+2)^2(m+3)^2}\right)^{1/2}$$

Due to the orthogonality property of the TWM, the accuracy of the first and second integrals can be obtained as follows

$$\left\|\int_0^x f(\tau)d\tau - \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\right\|^2 = \int_0^1 \left|\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} h_{nm}L_{nm}(X) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} h_{nm}L_{nm}(x)\right|^2 dx.$$

This implies that

$$\begin{aligned}\gamma_{nm}^2 &\le \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|h_{nm}|^2\int_0^1\int_0^x J_{nm}^2(\tau)d\tau dx\\ \gamma_{nm} &\le R\left(\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\frac{2m+1}{n^5(m+1)^2(m+2)^2(m+3)^2}\right)^{1/2}, \qquad (4.7)\end{aligned}$$

Where $t = \left(2^{k-1}\tau - n + 1\right)$. The second integral form of the TWM is bounded by

$$\left\|\int_0^x\int_0^\tau f(u)dud\tau - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} h_{nm}L_{nm}\right\|^2 =$$

$$\begin{aligned}\gamma_{nm}^2 &\le \int_0^1 \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}|h_{nm}|^2\int_0^x\int_0^\tau J_{nm}^2(u)dud\tau dx\\ \gamma_{nm} &\le R\left(\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\frac{2m+1}{n^5(m+1)^2(m+2)^2(m+3)^2}\right)^{1/2} \qquad (4.8)\end{aligned}$$

□

4.2. The final convergence result of the proposed method

In some of the above define error estimations, Section 4.1 is ending up with Theorem 2 which declares the convergence result of the above proposed numerical method with the consistency and stability of this article.

Theorem 4.3. *Let* $u \in L^2[0,1] \cap \rfloor[0,T]$ *be the exact solution of the given initial-boundary problem which is written in Equations* $(1.1) - (1.4)$*.Let us suppose that U(x,t) is the numerical solution of the given problem obtained by the proposed method.Let* $\Delta t = \frac{T}{N_t}$ *where* N_t *is the number of discretizations of the time interval.The proposed method is convergent in the sense that*

$$\|u(x,t_s) - U(x,t_s)\| \le \|u(x,0) - U(x,0)\| + s\beta\Delta t.$$

Notice that $\beta = \omega\gamma_{nm}$, $\omega \in \mathbb{R}$ *where* γ_{nm} *represents the error estimation of the given wavelet methods.Moreover, r represents the time step for* $s = 1, 2, ..., N_t$

Proof. From the previous analysis, recall the numerical solution and the exact solution at $t = t_{s+1}$, respectively,

$$U(x,t_{s+1}) = (t_{s+1}-t_s)\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} h_{nm}(L_{nm}(x) - x \otimes L_{nm}(1)) + U(x,t_s) + bc(t_s^{s+1},x), \tag{4.9}$$

and

$$u(x,t_{s+1}) = (t_{s+1}-t_s)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} h_{nm}(L_{nm}(x) - x \otimes L_{nm}(1)) + U(x,t_s) + bc(t_s^{s+1},x), \tag{4.10}$$

where $bc(t_s^{s+1},x) = f_0(t_{s+1}) - f_0(t_s) + x\,(f_1(t_{s+1} - f_1(t_s) - f_0(t_{s+1}) + f_0(t_s)).$ $\square$

Subtracting Equation (4.10) from Equation (4.11), the local error has been defined as follows:

$$\begin{aligned} \left|U\left(x,t_{s+1}\right) - u\left(x,t_s\right)\right|, \quad &\leq \quad \Delta t \sum_{n=2^{k-1}+1}^{\infty} \sum_{M}^{\infty} \left|h_{nm}(L_{nm}(x) - x \otimes L_{nm}(1))\right| \\ &\quad + \left|U\left(x,t_s\right) - u(x,t_s)\right|, \end{aligned} \tag{4.11}$$

Where $\Delta t = (t_{s+1} - t_s)$. Notice that the terms of $bc(t_s^{s+1},x)$ are extracted from the exact solution; then, there is no contribution of $bc(t_s^{s+1},x)$ on the error estimation. Moreover, equation (4.11) gives a strong connection between the convergence result of the proposed method and error estimations of wavelet methods. By defining $p_j = \left|U(x,t_j) - u(x,t_{j-1})\right|,\ j = 1,2,....,s+1$, we have

$$p_s \leq p_{s-1} + \beta\Delta t, \tag{4.12}$$

where

$$\beta = \begin{cases} \underbrace{\omega\left(\displaystyle\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty} \frac{2m+1}{n^5(m+1)^2(m+2)^2(m+3)^2}\right)^{1/2}}_{\gamma_{nm}}, & \text{for TWM;} \end{cases} \tag{4.13}$$

Here, ω is a constant dependent on R and $\max_{1\leq i\leq 2^{k-1}M} x_i$. Equation (4.12) and Equation (4.13) provides that the proposed method is consistent. The controllability of the error in this article can easily be seen by induction as follows:

$$\begin{aligned} p_1 &\leq p_0 + \beta\Delta t, && (4.14)\\ p_2 &\leq p_1 + \beta\Delta t \leq p_0 + 2\beta\Delta t, && (4.15)\\ &\vdots\ \ \vdots\ \ \vdots && (4.16)\\ p_{N_t} &\leq p_0 + N_t\beta\Delta t. && (4.17) \end{aligned}$$

Moreover, numerical solution is obtained by the initial condition of the equation, that is, $p_0 = 0$. From the definition of Δt, we can say that $N_t \Delta t = T$. This method is stable. It is important to recall that $\beta = \omega\gamma_{nm} \to 0$ as k and M increase. Consistency and stability, can be concluded that $p_{N_t} \to 0$ confirm the convergence of the proposed method.

5. NUMERICAL RESULTS AND DISCUSSION

Example 1.

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + X(t)u(x,t) - xsin(t) - txsin(t), \; 0 \le x \le 1, 0 < t \le T, \tag{5.1}$$

with initial condition

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = x, \qquad 0 \le x \le 1, \tag{5.2}$$

and Dirichlet boundary conditions

$$u(0,t) = 0, \qquad 0 < t \le T, \tag{5.3}$$

$$u(1,t) = sin(t), \qquad 0 < t \le T, \tag{5.4}$$

subject to the overspecified condition

$$u(0.5,t) = \frac{sin(t)}{2}, \qquad 0 < t \le T, \tag{5.5}$$

The exact solution is

$$u(x,t) = xsin(t),$$

and

$$X(t) = t.$$

x	Pointwise absolute error
	TWM
	$\Delta t = 10^{-3}$
	$k = 3, m = 4.$
0.0625	2.429 $\times 10^{-17}$
0.1250	4.857 $\times 10^{-17}$
0.1875	4.302 $\times 10^{-16}$
0.2500	9.714 $\times 10^{-17}$
0.3125	3.608 $\times 10^{-16}$
0.3750	8.604 $\times 10^{-16}$
0.4375	2.220 $\times 10^{-16}$
0.5000	1.943 $\times 10^{-16}$
0.5625	1.665 $\times 10^{-16}$
0.6250	7.216 $\times 10^{-16}$
0.6875	2.220 $\times 10^{-16}$
0.7500	1.665 $\times 10^{-15}$
0.8125	4.441 $\times 10^{-16}$
0.8750	4.441 $\times 10^{-16}$
0.9375	0000
1.000	3.886 $\times 10^{-16}$

In the above table, we have shown pointwise absolute error in the numerical solutions obtained by Taylor wavelet method (TWM).It is observed that at very less k and m, we are obtaining very good results. The errors are of order 10^{-16}.

Pointwise absolute error		
t	Exact X	TWM
0.1	0.1	9.931 $\times 10^{-4}$
0.2	0.2	9.925 $\times 10^{-4}$
0.3	0.3	9.980 $\times 10^{-4}$
0.4	0.4	9.915 $\times 10^{-4}$
0.5	0.5	9.956 $\times 10^{-4}$
0.6	0.6	0.018 $\times 10^{-3}$
0.7	0.7	0.086 $\times 10^{-3}$
0.8	0.8	0.032 $\times 10^{-3}$
0.9	0.9	0.033 $\times 10^{-3}$
1.0	1.0	0.047 $\times 10^{-3}$

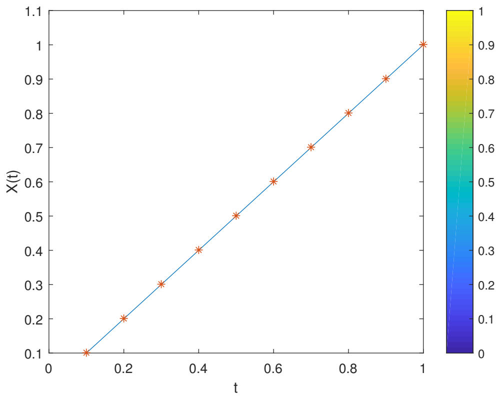

Figure 1: $X(t)$ and t

In Figure 1, We obtain exact and Taylor wavelet solution for X for $\Delta t = 10^{-3}$.

Example 2.

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + X(t)u(x,t) - xe^t - t^2 xe^t,\ 0 \leq x \leq 1, 0 < t \leq T, \tag{5.6}$$

with initial condition

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = x \qquad 0 \leq x \leq 1, \tag{5.7}$$

and Dirichlet boundary conditions

$$u(0,t) = 0, \qquad 0 < t \leq T, \tag{5.8}$$

$$u(1,t) = e^t, \qquad 0 < t \leq T, \tag{5.9}$$

subject to the overspecified condition

$$u(0.5,t) = \frac{e^t}{2}, \qquad 0 < t \leq T, \tag{5.10}$$

the exact solution is

$$u(x,t) = xe^t, \tag{5.11}$$

and

$$X(t) = 1 + t^2. \tag{5.12}$$

x	Pointwise absolute error
	TWM
	$\Delta t = 10^{-3}$
	$k = 3, m = 4.$
0.0625	2.776×10^{-17}
0.1250	5.551×10^{-17}
0.1875	3.275×10^{-15}
0.2500	1.110×10^{-16}
0.3125	3.331×10^{-15}
0.3750	6.550×10^{-15}
0.4375	3.109×10^{-15}
0.5000	2.220×10^{-16}
0.5625	5.107×10^{-15}
0.6250	6.661×10^{-15}
0.6875	3.775×10^{-15}
0.7500	1.310×10^{-14}
0.8125	2.887×10^{-15}
0.8750	4.885×10^{-15}
0.9375	3.997×10^{-15}
1.000	4.441×10^{-16}

In the above table, we have shown pointwise absolute error in the numerical solutions obtained by Taylor wavelet method(TWM). It is observed that at very less k and m, we are obtaining very good results. The errors are of order 10^{-15}.

Pointwise absolute error		
t	Exact X	TWM
0.1	1.01	0.241×10^{-3}
0.2	1.04	0.211×10^{-3}
0.3	1.09	0.161×10^{-3}
0.4	1.16	0.091×10^{-3}
0.5	1.25	0.001×10^{-3}
0.6	1.36	0.108×10^{-3}
0.7	1.49	0.228×10^{-3}
0.8	1.64	0.388×10^{-3}
0.9	1.81	0.558×10^{-3}
1.0	2.00	0.748×10^{-3}

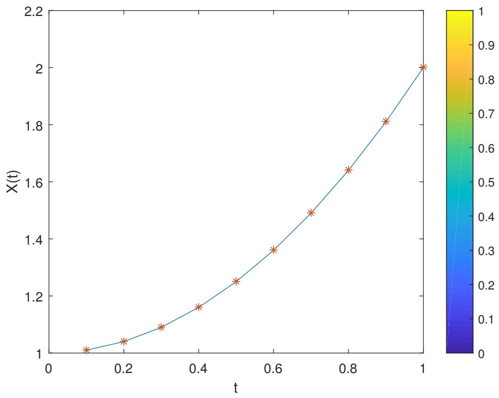

Figure 2: $X(t)$ and t

In Figure 2, We obtain exact and Taylor wavelet solution for X for $\Delta t = 10^{-3}$.

Example 3.

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + X(t)u(x,t) - (2xsin(t) + 2xcos(t) + txsin(t) + txcos(t)),\ 0 \le x \le 1, 0 < t \le T, \tag{5.13}$$

with initial condition

$$u(x,0) = x, \qquad \frac{\partial u}{\partial t}(x,0) = x, \qquad 0 \le x \le 1, \tag{5.14}$$

and Dirichlet boundary conditions

$$u(0,t) = 0, \qquad 0 < t \le T, \tag{5.15}$$

$$u(1,t) = sin(t) + cos(t), \qquad 0 < t \le T, \tag{5.16}$$

subject to overspecified condition

$$u(0.5,t) = \frac{(sin(t) + cos(t))}{2}, \qquad 0 < t \le T, \tag{5.17}$$

the exact solution is

$$u(x,t) = x(sin(t) + cos(t)), \tag{5.18}$$

and

$$X(t) = (1+t). \tag{5.19}$$

x	Pointwise absolute error
	TWM
	$\Delta t = 10^{-3}$
	$k = 3, m = 4.$
0.0625	2.776 $\times 10^{-17}$
0.1250	5.551 $\times 10^{-17}$
0.1875	3.275 $\times 10^{-15}$
0.2500	1.110 $\times 10^{-16}$
0.3125	3.331 $\times 10^{-15}$
0.3750	6.550 $\times 10^{-15}$
0.4375	3.109 $\times 10^{-15}$
0.5000	2.220 $\times 10^{-16}$
0.5625	5.107 $\times 10^{-15}$
0.6250	6.661 $\times 10^{-15}$
0.6875	3.553 $\times 10^{-15}$
0.7500	1.310 $\times 10^{-14}$
0.8125	2.887 $\times 10^{-15}$
0.8750	6.217 $\times 10^{-15}$
0.9375	2.887 $\times 10^{-15}$
1.000	4.441 $\times 10^{-16}$

In the above table, we have shown pointwise absolute error in the numerical solutions obtained by Taylor wavelet method(TWM). It is observed that at very less $k$and m, we are obtaining very good results. The errors are of order 10^{-15}.

Pointwise absolute error		
t	Exact X	TWM
0.1	1.1	0.001 $\times 10^{-3}$
0.2	1.2	0.181 $\times 10^{-3}$
0.3	1.3	0.271 $\times 10^{-3}$
0.4	1.4	0.361 $\times 10^{-3}$
0.5	1.5	0.451 $\times 10^{-3}$
0.6	1.6	0.541 $\times 10^{-3}$
0.7	1.7	0.631 $\times 10^{-3}$
0.8	1.8	0.721 $\times 10^{-3}$
0.9	1.9	0.811 $\times 10^{-3}$
1.0	2.0	9.999 $\times 10^{-4}$

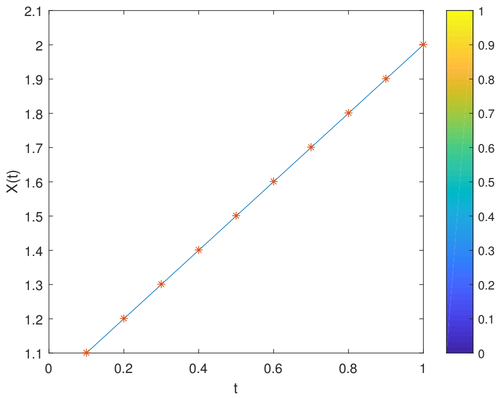

Figure 3: $X(t)$ and t

In Figure 3, We obtain exact and Taylor wavelet solution for X for $\Delta t = 10^{-3}$.

6. CONCLUSION

We have developed highly efficient and accurate numerical method based on Taylor wavelet for control parameter identification in the hyperbolic inverse problem. Using the uniform convergence property possessed by Taylor wavelet, we have derived a rigorous convergence analysis. Considering some collocation points in the domain, we get very good accuracy. The future extension of our work in the direction to taylor wavelet based collocation method for 2D and 3D hyperbolic inverse problems. We are also extending our method for paraboic inverse problem in higher dimensions.

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